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A SIMPLIFIED ANALYTICAL MODEL OF ROLLING/SLIDING BEHAVIOR AND FRICTION IN FOUR-POINT-CONTACT BALL BEARINGS AND SCREWS

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ABSTRACT

Four-point contact between ball and raceways is common in machine elements like ball bearings and ball screws. The ideal four-point-contact machine element is designed with pure rolling (i.e., no sliding at contact points) to minimize friction. However, this ideal may not always be achieved, leading to sliding and higher frictional forces. In this paper, a simplified analytical model for rolling/sliding behavior and friction in four-point contact is developed, based on Coulomb friction model and rigid body assumption. It is found that pure rolling is only possible when the contact-point geometry satisfies a certain relationship. When pure rolling condition fails to hold, the sliding contact point(s) can be determined analytically as a function of contact forces and contact angles. Case studies are presented to demonstrate how the proposed model could elucidate the roles of misalignments, manufacturing errors and loading conditions on rolling/sliding behavior and friction.

1. INTRODUCTION

Rolling element machine components such as rotary and linear ball bearings and ball screws are used in a wide range of machines to reduce friction. These machine components are commonly designed with four-point contact between ball and raceways because it offers increased rigidity and load capacity in a compact configuration [1]. Four-point contact is also adopted to reduce backlash and meet dynamic requirements of machine components [2]. Typically, four-point contact is achieved by placing oversized balls in Gothic-arch-type grooves or by using split raceways [1].

The rolling/sliding behavior at the contact interface of ball and grooves is very important in determining the friction behavior of four-point-contact machine elements. Sliding is characterized by non-zero relative linear velocity between the two surfaces at the contact point [3]; while rolling means there is no relative linear velocity at the contact interface. Sliding friction loss is typically 10^2–10^3 larger than that of rolling for metal material and parts [4–6]. Therefore, pure rolling is desired to minimize friction loss. However, imperfections such as manufacturing errors and misalignments, together with external loading, can induce sliding at some contact points, causing large increases in friction [7]. It is therefore important to understand when and how sliding occurs at contact points.

The typical modeling process for rolling/sliding behavior and friction of rolling elements is achieved by using comprehensive numerical models [7–9]. In such models, the ball motion is assumed, and the relative linear and angular velocity field between each ball and raceways is established over the contact area. Accordingly, the total frictional force and moment are obtained by integrating infinitesimal frictional forces as a function of the velocity field and contact stress distribution over the contact area. Static equilibrium is established including friction for each ball to back solve the ball motion which is compared to the assumed ball motion; the process is repeated iteratively until the solution converges. Though comprehensive, such iterative numerical solutions are computationally expensive and do not yield general insights into conditions that determine rolling/sliding of balls.

As an alternative to comprehensive models, simplified models have also been developed for studying rolling/sliding behavior and friction of balls in machine elements. For example, Jones [10] developed the so-called “race control theory” that requires one of the contacts in a two-point bearing

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to maintain pure rolling while spinning occurs only at the other contact. In his model, details about the contact area are ignored, hence frictional forces are only functions of the friction coefficient, normal contact force and relative velocity at the contact point. These simplifying assumptions are based on the experimental results and are proved to be valid for bearings under normal operating conditions. Halpin and Tran [1] replace Jones' "raceway control theory" with a minimum energy criterion to allow its use in four-point contact. However, though simplified, these models [1,10] do not yield analytical solutions; hence they do not provide insights into rolling/sliding behavior and the resultant frictional forces.

The key contributions of this paper are in: (1) proposing a simplified analytical model for rolling/sliding behavior and friction in four-point-contact rolling-ball machine elements, and (2) deriving insights from the model that could be useful for design and analysis of four-point-contact rolling-ball machine elements. The paper is organized as follows. In Section 2, rigid body assumptions are made about the balls, leading to point instead of area contact. By adopting the Coulomb friction model with steady-state dynamic conditions, analytical conditions for rolling/sliding behavior of balls are derived, along with the resultant frictional forces and power loss. Insights emanating from the proposed analytical model, which could be very useful for design and analysis of four-point-contact rolling-ball machine elements, are also discussed. Section 3 presents two case studies that demonstrate how the proposed model could shed light on the roles of misalignments, manufacturing errors and loading conditions on rolling/sliding behavior and friction. Finally, conclusions and future work are presented in Section 4.

2. ROLLING/SLIDING MODELING FOR FOUR-POINT CONTACT

There are many different forms of four-point contact in machine components such as ball bearings, linear guides and ball screws. But four-point contact is similar in all these components. Without loss of generality, Fig. 1 shows the basic configuration where a ball is in four-point point contact with two linear rails. It can be generalized to ball bearing or ball screw applications with circular or helical grooves instead of linear rails. In the setup, the bottom groove is fixed and the top groove is moving at constant velocity of magnitude v.

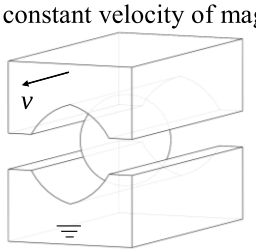

Figure 1. Basic configuration of four-point contact

As our assumption, the contact area is represented as a point, on which normal contact force and frictional force are concentrated. Normal contact forces as a result of external loading, are calculated a priori based on their own static equilibrium. Frictional forces based on Coulomb friction law are assumed not to affect the normal contact forces. The basic idea is to establish static equilibrium of force and moment for the friction generated due to rolling/sliding behavior, and back solve the assumed ball motion.

2.1. Rigid Body Kinematics

Friction is a function of ball motion, it is highly dependent on the rolling/sliding behavior of the ball. Here the kinematics of the four-point-contact ball is first introduced. Figure 2 depicts the cross section of the basic configuration shown in Fig. 1. Let us define a global coordinate system (CS={x, y, z}), fixed in space with its z-axis passing through ball center and its xy-plane parallel to the cross section. A ball, with radius R_B, is in four-point contact with the two grooves at BL, BR, TR and TL (representing Bottom/Top and Left/Right) points. The four-point contact happens at the cross section with contact angles β_{BL}, β_{BR}, β_{TR} and β_{TL} respectively measured from $\pm y$-axis (see Fig. 2), so β_{BL}, β_{BR}, β_{TR} and $\beta_{TL} \in (0, \pi/2)$. Local coordinate systems CS_{BL}, CS_{BR}, CS_{TR} and CS_{TL} are established for the corresponding contact points such that local z-axes are parallel to global z-axis and local y-axes point to the origin of global coordinate system CS as shown in Fig. 2. The movement of top groove in global z-direction at constant velocity v induces movement of the ball through frictional forces. Assume at steady state, the ball translates with linear velocity v_B in global z-direction (i.e., $\mathbf{v}_B=\{0, 0, v_B\}^T$ in vector form) and rotates with $\mathbf{\Omega}$ ($=\{\omega_x, \omega_y, \omega_z\}^T$) about an axis passing through the ball center.

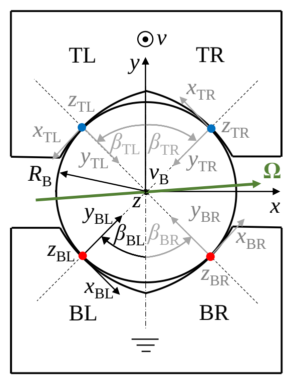

Figure 2. Geometry, global and local coordinate systems, and kinematic variables for the four-point contact

Linear velocities at the contact points on both ball side and groove side can be expressed based on rigid body kinematics. Focusing on the BL contact point, $\mathbf{q}_{BL}$ is defined as the vector from the ball center to the BL contact point and is given in global coordinate system CS as

$$\mathbf{q}_{BL} = \begin{Bmatrix} -R_B \sin\beta_{BL} \\ -R_B \cos\beta_{BL} \\ 0 \end{Bmatrix} \tag{1}$$

The linear velocity at the BL contact point on the ball side in CS can be expressed as

$$\mathbf{v}_{BL} = \mathbf{\Omega} \times \mathbf{q}_{BL} + \mathbf{v}_B = \begin{Bmatrix} \omega_z R_B \cos\beta_{BL} \\ -\omega_z R_B \sin\beta_{BL} \\ v_B - \omega_x R_B \cos\beta_{BL} + \omega_y R_B \sin\beta_{BL} \end{Bmatrix} \tag{2}$$

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Since the bottom groove is fixed, linear velocity at the BL contact point on the groove side is zero. Thus the relative velocity at the BL contact point is

$$\Delta\mathbf{v}_{\mathrm{BL}} = \mathbf{v}_{\mathrm{BL}} - \mathbf{0} = \mathbf{v}_{\mathrm{BL}} \tag{3}$$

The relative velocity expressed in local coordinate system $\mathrm{CS}_{\mathrm{BL}}$ can be formulated as

$$\begin{aligned}(\Delta\mathbf{v}_{\mathrm{BL}})_{\mathrm{BL}} &= \begin{Bmatrix}(\Delta\mathbf{v}_{\mathrm{BL}})_{\mathrm{BL},x}\\(\Delta\mathbf{v}_{\mathrm{BL}})_{\mathrm{BL},y}\\(\Delta\mathbf{v}_{\mathrm{BL}})_{\mathrm{BL},z}\end{Bmatrix} = \mathbf{T}^{\mathrm{T}}_{\text{CS-CS}_{\mathrm{BL}}}\Delta\mathbf{v}_{\mathrm{BL}}\\ &= \begin{Bmatrix}\omega_z R_{\mathrm{B}}\\0\\v_{\mathrm{B}} - \omega_x R_{\mathrm{B}}\cos\beta_{\mathrm{BL}} + \omega_y R_{\mathrm{B}}\sin\beta_{\mathrm{BL}}\end{Bmatrix}\end{aligned} \tag{4}$$

where $\mathbf{T}_{\text{CS-CS}_{\mathrm{BL}}}$ is the transformation matrix from $\mathrm{CS}_{\mathrm{BL}}$ to CS given by

$$\mathbf{T}_{\text{CS-CS}_{\mathrm{BL}}} = \begin{bmatrix}\cos\beta_{\mathrm{BL}} & \sin\beta_{\mathrm{BL}} & 0\\ -\sin\beta_{\mathrm{BL}} & \cos\beta_{\mathrm{BL}} & 0\\ 0 & 0 & 1\end{bmatrix} \tag{5}$$

The local x- and z- components of the relative velocity in Eq. (4) are of interest because they determine the sliding/no sliding state of the contact point. If they are both zero, there is no sliding at the contact point; if any of them is non-zero, the contact point is sliding. Following the same procedure, the relative velocities at other contact points can be derived as

$$\begin{aligned}&\begin{Bmatrix}(\Delta\mathbf{v}_{\mathrm{BR}})_{\mathrm{BR},x}\\(\Delta\mathbf{v}_{\mathrm{BR}})_{\mathrm{BR},z}\end{Bmatrix} == \begin{Bmatrix}\omega_z R_{\mathrm{B}}\\ v_{\mathrm{B}} - \omega_x R_{\mathrm{B}}\cos\beta_{\mathrm{BR}} - \omega_y R_{\mathrm{B}}\sin\beta_{\mathrm{BR}}\end{Bmatrix},\\ &\begin{Bmatrix}(\Delta\mathbf{v}_{\mathrm{TR}})_{\mathrm{TR},x}\\(\Delta\mathbf{v}_{\mathrm{TR}})_{\mathrm{TR},z}\end{Bmatrix} = \begin{Bmatrix}\omega_z R_{\mathrm{B}}\\ v_{\mathrm{B}} - v + \omega_x R_{\mathrm{B}}\cos\beta_{\mathrm{TR}} - \omega_y R_{\mathrm{B}}\sin\beta_{\mathrm{TR}}\end{Bmatrix},\\ &\begin{Bmatrix}(\Delta\mathbf{v}_{\mathrm{TL}})_{\mathrm{TL},x}\\(\Delta\mathbf{v}_{\mathrm{TL}})_{\mathrm{TL},z}\end{Bmatrix} = \begin{Bmatrix}\omega_z R_{\mathrm{B}}\\ v_{\mathrm{B}} - v + \omega_x R_{\mathrm{B}}\cos\beta_{\mathrm{TL}} + \omega_y R_{\mathrm{B}}\sin\beta_{\mathrm{TL}}\end{Bmatrix}\end{aligned} \tag{6}$$

Observe that the local x-components of relative velocities for the four contact points are all the same as given by $\omega_z R_{\mathrm{B}}$. Ideally pure rolling (i.e., no sliding at all four contact points) is desired so that friction loss is minimized. In order for pure rolling to exist, ω_z=0 has to be enforced. The following subsections mainly deal with ω_z=0. The case of $\omega_z \neq 0$ will be examined at the end of this section.

2.2. Condition for Pure Rolling (No Sliding)

Since ω_z=0 is enforced, the local x-components of relative velocities in Eqs. (4) and (6) are all zero. The local z-components of relative velocities can be stacked together as

$$\begin{Bmatrix}\Delta\mathbf{v}_{\mathrm{BL},z}\\ \Delta\mathbf{v}_{\mathrm{BR},z}\\ \Delta\mathbf{v}_{\mathrm{TR},z}\\ \Delta\mathbf{v}_{\mathrm{TL},z}\end{Bmatrix} = \underbrace{\begin{bmatrix}1 & -R_{\mathrm{B}}\cos\beta_{\mathrm{BL}} & R_{\mathrm{B}}\sin\beta_{\mathrm{BL}}\\ 1 & -R_{\mathrm{B}}\cos\beta_{\mathrm{BR}} & -R_{\mathrm{B}}\sin\beta_{\mathrm{BR}}\\ 1 & R_{\mathrm{B}}\cos\beta_{\mathrm{TR}} & -R_{\mathrm{B}}\sin\beta_{\mathrm{TR}}\\ 1 & R_{\mathrm{B}}\cos\beta_{\mathrm{TL}} & R_{\mathrm{B}}\sin\beta_{\mathrm{TL}}\end{bmatrix}}_{A_v}\begin{Bmatrix}v_{\mathrm{B}}\\ \omega_x\\ \omega_y\end{Bmatrix} - \begin{Bmatrix}0\\0\\v\\v\end{Bmatrix} \tag{7}$$

Notice here because the local and global z-axes are aligned, the relative velocities in local and global CS are the same. Again, pure rolling is desired, which means

$$\begin{Bmatrix}\Delta\mathbf{v}_{\mathrm{BL},z}\\ \Delta\mathbf{v}_{\mathrm{BR},z}\\ \Delta\mathbf{v}_{\mathrm{TR},z}\\ \Delta\mathbf{v}_{\mathrm{TL},z}\end{Bmatrix} = (A_v)_{4\times 3}\begin{Bmatrix}v_{\mathrm{B}}\\ \omega_x\\ \omega_y\end{Bmatrix} - \begin{Bmatrix}0\\0\\v\\v\end{Bmatrix} = \begin{Bmatrix}0\\0\\0\\0\end{Bmatrix} \tag{8}$$

It is an overdetermined system, so solution only exists when A_v satisfies the condition:

$$\begin{aligned}&\sin(\beta_{\mathrm{BL}} + \beta_{\mathrm{TL}}) + \sin(\beta_{\mathrm{BR}} - \beta_{\mathrm{TL}})\\ &= \sin(\beta_{\mathrm{BR}} + \beta_{\mathrm{TR}}) + \sin(\beta_{\mathrm{BL}} - \beta_{\mathrm{TR}})\end{aligned} \tag{9}$$

or

$$\frac{\cos\beta_{\mathrm{BL}} - \cos\beta_{\mathrm{BR}}}{\sin\beta_{\mathrm{BL}} + \sin\beta_{\mathrm{BR}}} = -\frac{\cos\beta_{\mathrm{TL}} - \cos\beta_{\mathrm{TR}}}{\sin\beta_{\mathrm{TL}} + \sin\beta_{\mathrm{TR}}} \tag{10}$$

Graphically, the relationship in Eq. (10) means the two lines passing through the two contact points on the same groove are parallel as shown in Fig. 3. The two lines also gather all the points with zero relative velocities, so they are defined as "zero velocity lines" in this paper.

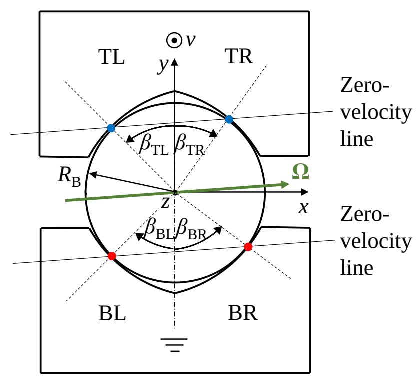

Figure 3. Graphical representation of pure rolling condition

If the condition in Eqs. (9) or (10) (i.e., Fig. 3) holds, the unique solution of ball motion for pure rolling in Eq. (8) is

$$\begin{Bmatrix}v_{\mathrm{B}}\\ \omega_x\\ \omega_y\end{Bmatrix} = \frac{v\begin{Bmatrix}\sin(\beta_{\mathrm{BL}} + \beta_{\mathrm{BR}})\\ 1/R_{\mathrm{B}}(\sin\beta_{\mathrm{BL}} + \sin\beta_{\mathrm{BR}})\\ 1/R_{\mathrm{B}}(\cos\beta_{\mathrm{BL}} - \cos\beta_{\mathrm{BR}})\end{Bmatrix}}{\sin(\beta_{\mathrm{BL}} + \beta_{\mathrm{BR}}) + \sin(\beta_{\mathrm{BL}} + \beta_{\mathrm{TL}}) + \sin(\beta_{\mathrm{BR}} - \beta_{\mathrm{TL}})} \tag{11}$$

It can be observed that $\omega_y/\omega_x=(\cos\beta_{\mathrm{BL}}-\cos\beta_{\mathrm{BR}})/(\sin\beta_{\mathrm{BL}}+\sin\beta_{\mathrm{BR}})$, the ratio is exactly the same as in Eq. (10), meaning that the rotating axis of the ball is also parallel to the two zero-velocity lines in Fig. 3. Ideally, when pure rolling happens, friction loss is zero because there is no relative velocity between the two mating surfaces at any contact point.

If the condition in Eqs. (9) or (10) (i.e., Fig. 3) fails to hold due to imperfections such as manufacturing errors, misalignments and/or external loading, pure rolling do not exist, and sliding must take place. An observation of A_v in Eq. (7) is rank(A_v)=3 (proof can be found in Annex A. 1), indicating that at least three equations in Eq. (7) can be set to 0. The

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physical meaning of it is three-point rolling is always possible kinematically. It also implies that if there is only two-point or three-point contact, pure rolling with zero friction loss is always possible.

Other than pure rolling (i.e., no sliding), there are many possible combinations of sliding and no sliding for the four contact points: one-point sliding, two-point sliding, three-point sliding and four-point sliding. The conditions and behaviors for these cases are presented in the following two subsections.

2.3. One-point Sliding

There are four possible configurations of one-point sliding in four-point contact, namely, any of the four contact points can be sliding. The questions are: (1) Can static equilibrium of frictional force and moment be established when rolling/sliding friction is taken into consideration (existence of solution)? (2) Is the solution unique?

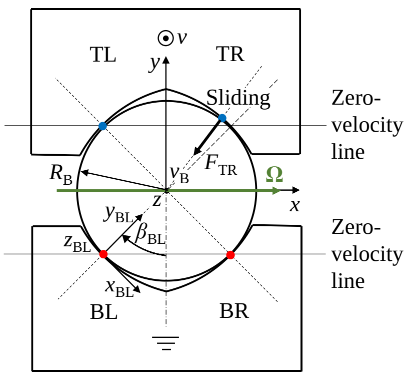

Figure 4. Graphical representation of TR sliding

Without loss of generality, let us assume that only the TR contact point is sliding. The graphical representation is shown in Fig. 4, the zero-velocity lines pass through the three non-sliding contact points. The TR contact point is deviated from the zero-velocity line, so its relative velocity is non-zero (i.e., there is sliding). The kinematic relationship for the non-sliding contact points is

$$\begin{Bmatrix} (\Delta\mathbf{v}_{BL})_{BL,z} \\ (\Delta\mathbf{v}_{BR})_{BR,z} \\ (\Delta\mathbf{v}_{TL})_{TL,z} \end{Bmatrix} = \bar{A}_v \begin{Bmatrix} v_B \\ \omega_x \\ \omega_y \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ v \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix},$$
$$\bar{A}_v = \begin{bmatrix} 1 & -R_B\cos\beta_{BL} & R_B\sin\beta_{BL} \\ 1 & -R_B\cos\beta_{BR} & -R_B\sin\beta_{BR} \\ 1 & R_B\cos\beta_{TL} & R_B\cos\beta_{TL} \end{bmatrix} \tag{12}$$

It has a unique solution with the same expression as Eq. (11), because rank($\bar{A}_v$)=3 (proof can be found in Annex A. 1). The relative velocity in the sliding contact point TR is

$$\Delta\mathbf{v}_{TR,z} = \frac{-C_\beta v}{\sin(\beta_{BL}+\beta_{BR})+\sin(\beta_{BL}+\beta_{TL})+\sin(\beta_{BR}-\beta_{TL})},$$
$$C_\beta = \sin(\beta_{BL}+\beta_{TL}) - \sin(\beta_{BR}+\beta_{TR}) - \sin(\beta_{BL}-\beta_{TR}) + \sin(\beta_{BR}-\beta_{TL}) \tag{13}$$

According to Coulomb friction law, it gives rise to a sliding friction in $+z$-direction with magnitude

$$f_{TR,z} = -\mu_k F_{TR}\,\mathrm{sgn}(\Delta\mathbf{v}_{TR,z}) \tag{14}$$

where μ_k is the kinetic friction coefficient and F_{TR} is the normal contact force. Put it in global coordinate system CS in vector form, it is

$$\mathbf{f}_{TR} = \mathbf{T}_{CS\text{-}CS_{TR}} \begin{Bmatrix} 0 \\ 0 \\ f_{TR,z} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ f_{TR,z} \end{Bmatrix} \tag{15}$$

The moment of $\mathbf{f}_{TR}$ about the ball center is

$$\mathbf{M}_{TR} = \mathbf{q}_{TR} \times \mathbf{f}_{TR} = \begin{Bmatrix} f_{TR,z}R_B\cos\beta_{TR} \\ -f_{TR,z}R_B\sin\beta_{TR} \\ 0 \end{Bmatrix} \tag{16}$$

The sliding friction of the TR contact point is determined in Eqs. (14)-(16) once ball motion is specified, but the rolling friction of the other three contact points are not. Let us assume the rolling friction in the BL contact point has components $f_{BL,x}$ and $f_{BL,z}$ in local coordinate system CS_{BL}. The vector form of the frictional force and moment in global coordinate system CS are obtained as

$$\mathbf{f}_{BL} = \mathbf{T}_{CS\text{-}CS_{BL}} \begin{Bmatrix} f_{BL,x} \\ 0 \\ f_{BL,z} \end{Bmatrix} = \begin{bmatrix} \cos\beta_{BL} & 0 \\ -\sin\beta_{BL} & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} f_{BL,x} \\ f_{BL,z} \end{Bmatrix} \tag{17}$$

$$\mathbf{M}_{BL} = \mathbf{q}_{BL} \times \mathbf{f}_{BL} = \begin{bmatrix} 0 & -R_B\cos\beta_{BL} \\ 0 & R_B\sin\beta_{BL} \\ R_B & 0 \end{bmatrix} \begin{Bmatrix} f_{BL,x} \\ f_{BL,z} \end{Bmatrix} \tag{18}$$

Similarly, $f_{BR,x}$ and $f_{BR,z}$, $f_{TL,x}$ and $f_{TL,z}$ are assumed for the BR and TL contact points, and $\mathbf{f}_{BR}$, $\mathbf{M}_{BR}$, $\mathbf{f}_{TL}$, $\mathbf{M}_{TL}$ are obtained following the same procedure in Eqs. (17) and (18). The static equilibrium of frictional force and moment for the ball is established as

$$\begin{cases} \sum \mathbf{f} = \mathbf{f}_{BL} + \mathbf{f}_{BR} + \mathbf{f}_{TR} + \mathbf{f}_{TL} = \mathbf{0} \\ \sum \mathbf{M} = \mathbf{M}_{BL} + \mathbf{M}_{BR} + \mathbf{M}_{TR} + \mathbf{M}_{TL} = \mathbf{0} \end{cases} \tag{19}$$

It has a unique solution given by

$$\begin{Bmatrix} f_{BL,x} \\ f_{BR,x} \\ f_{TL,x} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix},$$
$$\begin{Bmatrix} f_{BL,z} \\ f_{BR,z} \\ f_{TL,z} \end{Bmatrix} = \frac{f_{TR,z}\mathbf{D}_\beta}{\sin(\beta_{BL}+\beta_{BR})+\sin(\beta_{BL}+\beta_{TL})+\sin(\beta_{BR}-\beta_{TL})}, \tag{20}$$
$$\mathbf{D}_\beta = \begin{Bmatrix} \sin(\beta_{BR}+\beta_{TR})+\sin(\beta_{TL}+\beta_{TR})-\sin(\beta_{BR}-\beta_{TL}) \\ -\sin(\beta_{BL}+\beta_{TL})-\sin(\beta_{TL}+\beta_{TR})+\sin(\beta_{BL}-\beta_{TR}) \\ -\sin(\beta_{BL}+\beta_{BR})-\sin(\beta_{BR}+\beta_{TR})-\sin(\beta_{BL}-\beta_{TR}) \end{Bmatrix}$$

The local x-components of the rolling frictions are all 0. Focusing on the non-zero local z-components, the signs of the frictional forces are: if the sliding friction $f_{TR,z}>0$, then $f_{BL,z}>0$, $f_{BR,z}<0$ and $f_{TL,z}<0$. The directions of the frictional forces in a contact point pair (BL and TR, BR and TL) are the same, they

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are different from those of the other contact point pair. It is required for the frictional force and moment to balance.

The rolling forces need to satisfy the following constraints according Coulomb friction law

$$\begin{aligned} \left|f_{\mathrm{BL},z}\right| &< \mu_{\mathrm{k}} F_{\mathrm{BL}}, \\ \left|f_{\mathrm{BR},z}\right| &< \mu_{\mathrm{k}} F_{\mathrm{BR}}, \\ \left|f_{\mathrm{TL},z}\right| &< \mu_{\mathrm{k}} F_{\mathrm{TL}} \end{aligned} \tag{21}$$

which gives

$$K_{\mathrm{TR}} < K_{\mathrm{BL}}, K_{\mathrm{BR}}, K_{\mathrm{TL}} \tag{22}$$

where K for each contact point is defined as

$$\begin{aligned} K_{\mathrm{BL}} &= \frac{F_{\mathrm{BL}}}{\sin(\beta_{\mathrm{BR}}+\beta_{\mathrm{TR}})+\sin(\beta_{\mathrm{TR}}+\beta_{\mathrm{TL}})+\sin(\beta_{\mathrm{TL}}-\beta_{\mathrm{BR}})}, \\ K_{\mathrm{BR}} &= \frac{F_{\mathrm{BR}}}{\sin(\beta_{\mathrm{BL}}+\beta_{\mathrm{TL}})+\sin(\beta_{\mathrm{TR}}+\beta_{\mathrm{TL}})+\sin(\beta_{\mathrm{TR}}-\beta_{\mathrm{BL}})}, \\ K_{\mathrm{TR}} &= \frac{F_{\mathrm{TR}}}{\sin(\beta_{\mathrm{BL}}+\beta_{\mathrm{BR}})+\sin(\beta_{\mathrm{BL}}+\beta_{\mathrm{TL}})+\sin(\beta_{\mathrm{BR}}-\beta_{\mathrm{TL}})}, \\ K_{\mathrm{TL}} &= \frac{F_{\mathrm{TL}}}{\sin(\beta_{\mathrm{BL}}+\beta_{\mathrm{BR}})+\sin(\beta_{\mathrm{BR}}+\beta_{\mathrm{TR}})+\sin(\beta_{\mathrm{BL}}-\beta_{\mathrm{TR}})} \end{aligned} \tag{23}$$

All K values are positive (proof can be found in Annex A. 1).

There is a characteristic parameter K associated with each contact point. Taking TR as example, K_{TR} is proportional to the normal contact F_{TR} and inversely proportional to the sum of three sinusoidal terms of the contact angles other than its own (β_{BL}, β_{BR} and β_{TL} but no β_{TR}). The parameter K for TR contact point needs to be smallest among the four for TR to be sliding as shown by inequality (22). The existence and uniqueness of one-point sliding is thus proved.

Remark 1. In fact, the condition for the TR contact point to be sliding in (22) can be generalized to other points sliding: sliding happens and only happens in the contact point whose characteristic parameter K given in Eq. (23) is the smallest among the four contact points.

In those contact points with no sliding, no friction loss is generated because there is no relative velocity at the contact interface. Friction loss only generates at the sliding contact point. Measured in power, it is

$$P_{f,\mathrm{X}} = K_{\mathrm{X}} v \mu_{\mathrm{k}} \left|C_\beta\right|, \quad \mathrm{X} \in \{\mathrm{BL}, \mathrm{BR}, \mathrm{TR}, \mathrm{TL}\} \tag{24}$$

where C_β is given in Eq. (13).

Remark 2. Although the sliding contact point is uniquely determined based on the static equilibrium of frictional force and moment and Coulomb friction law, the friction loss is always minimum because sliding happens at the contact point with the smallest K.

2.4. General Rolling/Sliding Cases

So far, the condition for pure rolling is established in Subsection 2.2. If pure rolling condition fails to hold and sliding has to occur, the existence and uniqueness of one-point sliding has been proved in Subsection 2.3. Conditions for general cases of four-point contact such as two-point sliding, three-point sliding and four-point sliding are shown in this subsection.

Let us first look at two-point sliding. There are 6 possible states (combinations of any two contact points from four). Without loss of generality, let us assume that the TR and TL contact points are sliding and there is no sliding at the BL and BR contact points. The kinematic relationship is

$$\begin{Bmatrix} \Delta \mathbf{v}_{\mathrm{BL},z} \\ \Delta \mathbf{v}_{\mathrm{BR},z} \end{Bmatrix} = \begin{bmatrix} 1 & -R_{\mathrm{B}}\cos\beta_{\mathrm{BL}} & R_{\mathrm{B}}\sin\beta_{\mathrm{BL}} \\ 1 & -R_{\mathrm{B}}\cos\beta_{\mathrm{BR}} & -R_{\mathrm{B}}\sin\beta_{\mathrm{BR}} \end{bmatrix} \begin{Bmatrix} v_{\mathrm{B}} \\ \omega_x \\ \omega_y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{25}$$

The set of equations is underdetermined, so there are infinitely many kinematic possibilities for two-point no sliding. The sliding velocities at the other two contact points are

$$\begin{Bmatrix} \Delta \mathbf{v}_{\mathrm{TR},z} \\ \Delta \mathbf{v}_{\mathrm{TL},z} \end{Bmatrix} = \begin{bmatrix} 1 & R_{\mathrm{B}}\cos\beta_{\mathrm{TR}} & -R_{\mathrm{B}}\sin\beta_{\mathrm{TR}} \\ 1 & R_{\mathrm{B}}\cos\beta_{\mathrm{TL}} & R_{\mathrm{B}}\sin\beta_{\mathrm{TL}} \end{bmatrix} \begin{Bmatrix} v_{\mathrm{B}} \\ \omega_x \\ \omega_y \end{Bmatrix} \tag{26}$$

Again the sliding frictions are in +z-direction with magnitude

$$\begin{aligned} f_{\mathrm{TR},z} &= -\mu_{\mathrm{k}} F_{\mathrm{TR}} \operatorname{sgn}(\Delta \mathbf{v}_{\mathrm{TR},z}), \\ f_{\mathrm{TL},z} &= -\mu_{\mathrm{k}} F_{\mathrm{TL}} \operatorname{sgn}(\Delta \mathbf{v}_{\mathrm{TL},z}) \end{aligned} \tag{27}$$

The rolling frictions in the BL and BR contact points are unknown. Following the same procedure in Eqs. (15)-(19). The frictional force and moment equilibrium for the ball can be established and solved. The condition for two-point sliding (TR and TL) turns out to be

$$\operatorname{sgn}(\Delta \mathbf{v}_{\mathrm{TR},z}) = -\operatorname{sgn}(\Delta \mathbf{v}_{\mathrm{TL},z}) \tag{28}$$

and

$$K_{\mathrm{TR}} = K_{\mathrm{TL}} < K_{\mathrm{BL}}, K_{\mathrm{BR}} \tag{29}$$

where the definition of K is the same as in Eq. (23).

The condition in Eq. (28) specifies that the directions of the relative velocities are different for the two sliding points. As shown in Fig. 5, the kinematic relationship for no sliding in Eq. (25) is represented as the bottom zero-velocity line. As long as the zero-velocity line for the top groove falls between the two limits, the condition in Eq. (28) is satisfied.

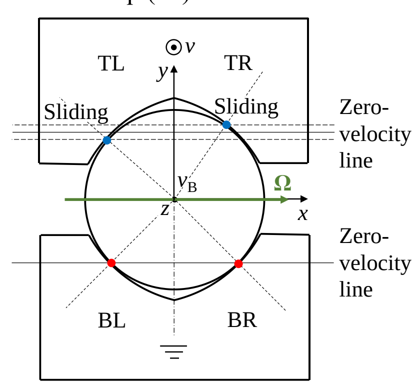

Figure 5. Graphical representation of TR and TL two-point sliding

For all possible kinematic states, the friction loss is

$$P_f = \left(aK_{\mathrm{TR}} + (1-a)K_{\mathrm{TL}}\right) v \mu_{\mathrm{k}} \left|C_\beta\right|, \; a \in [0 \;\; 1] \tag{30}$$

Since K_{TR}=K_{TL}, friction loss P_f is the same for all possible kinematic states. Thus minimum energy principal does not help uniquely determine the state of the system. Although in reality,

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such strict condition as $K_{TR}=K_{TL}$ is not likely to happen, theoretically other criteria need to be came up with to uniquely determine ball motion.

More generally, conditions for three-point sliding (assuming BR, TR and TL sliding) are

$$-\operatorname{sgn}\left(\Delta\mathbf{v}_{\mathrm{BR},z}\right)=\operatorname{sgn}\left(\Delta\mathbf{v}_{\mathrm{TR},z}\right)=-\operatorname{sgn}\left(\Delta\mathbf{v}_{\mathrm{TL},z}\right) \tag{31}$$

and

$$K_{\mathrm{BR}}=K_{\mathrm{TR}}=K_{\mathrm{TL}}<K_{\mathrm{BL}} \tag{32}$$

Conditions for all four-point sliding are

$$\begin{aligned}&\operatorname{sgn}\left(\Delta\mathbf{v}_{\mathrm{BL},z}\right)=-\operatorname{sgn}\left(\Delta\mathbf{v}_{\mathrm{BR},z}\right)\\&=\operatorname{sgn}\left(\Delta\mathbf{v}_{\mathrm{TR},z}\right)=-\operatorname{sgn}\left(\Delta\mathbf{v}_{\mathrm{TL},z}\right)\end{aligned} \tag{33}$$

and

$$K_{\mathrm{BL}}=K_{\mathrm{BR}}=K_{\mathrm{TR}}=K_{\mathrm{TL}} \tag{34}$$

Again there are infinitely many possible kinematic states for three-point sliding and four-point sliding, and friction loss for all the states are the same.

Overall, rolling/sliding behavior of the four-point-contact ball all depends on the relative magnitude of the characteristic parameter K for each contact point defined in Eq. (23). Sliding happens and only happens in the contact point(s) whose characteristic parameter K is (are) smallest. When some of K are equal, multiple-point sliding can take place at these contact points.

So far, the analysis of rolling and sliding are based on the condition $\omega_z=0$. The case of $\omega_z\neq0$ is not feasible because there are relative sliding velocities in local x-direction as shown in Eqs. (4) and (6), thus frictional moment about global z-axis is non-zero and static equilibrium can not be established.

3. CASE STUDIES

Based on the proposed analytical model in the preceding section, case studies are presented in this section to show the effect of misalignments, manufacturing errors and external loading on friction behavior of four-point contact, particularly the rolling/sliding states, ball motion and friction loss.

3.1. Normal Contact Force Model

It is assumed in the proposed model that normal contact forces as a result of external loading can be determined a priori based on their own static equilibrium. The frictional force and moment establish static equilibrium, and do not affect the normal contact forces and the contact angles. The calculation of normal contact forces given β is briefly summarized here. Assuming external forces N_x and N_y are exerted on the top groove which has a misalignment angle θ as shown in Fig. 6, static equilibrium of the ball and the top groove gives

$$\underbrace{\begin{bmatrix}\sin\beta_{\mathrm{BL}} & -\sin\beta_{\mathrm{BR}} & -\sin\beta_{\mathrm{TR}} & \sin\beta_{\mathrm{TL}}\\ \cos\beta_{\mathrm{BL}} & \cos\beta_{\mathrm{BR}} & -\cos\beta_{\mathrm{TR}} & -\cos\beta_{\mathrm{TL}}\\ 0 & 0 & -\sin\beta_{\mathrm{TR}} & \sin\beta_{\mathrm{TL}}\\ 0 & 0 & -\cos\beta_{\mathrm{TR}} & -\cos\beta_{\mathrm{TL}}\end{bmatrix}}_{A_N}\begin{Bmatrix}F_{\mathrm{BL}}\\F_{\mathrm{BR}}\\F_{\mathrm{TR}}\\F_{\mathrm{TL}}\end{Bmatrix}=\begin{Bmatrix}0\\0\\N_x\\-N_y\end{Bmatrix} \tag{35}$$

Since rank(A_N)=4, Eq. (35) always has a unique solution

$$\begin{Bmatrix}F_{\mathrm{BL}}\\F_{\mathrm{BR}}\\F_{\mathrm{TR}}\\F_{\mathrm{TL}}\end{Bmatrix}=\begin{Bmatrix}-\left(N_x\cos\beta_{\mathrm{BR}}-N_y\sin\beta_{\mathrm{BR}}\right)/\sin\left(\beta_{\mathrm{BL}}+\beta_{\mathrm{BR}}\right)\\ \left(N_x\cos\beta_{\mathrm{BL}}+N_y\sin\beta_{\mathrm{BL}}\right)/\sin\left(\beta_{\mathrm{BL}}+\beta_{\mathrm{BR}}\right)\\ -\left(N_x\cos\beta_{\mathrm{TL}}-N_y\sin\beta_{\mathrm{TL}}\right)/\sin\left(\beta_{\mathrm{TL}}+\beta_{\mathrm{TR}}\right)\\ \left(N_x\cos\beta_{\mathrm{TR}}+N_y\sin\beta_{\mathrm{TR}}\right)/\sin\left(\beta_{\mathrm{TL}}+\beta_{\mathrm{TR}}\right)\end{Bmatrix} \tag{36}$$

Negative solution for the normal contact forces means loss of contact at those contact points. As a result, there will be three-point or two-point contact. In such situation, the ball can always operate in pure rolling with zero friction loss. In order for four-point contact to take place (i.e., the normal contact force terms in Eq. (36) are all positive), N_x and N_y need to satisfy

$$\begin{cases}N_y>0\\-\min\left(\tan\beta_{\mathrm{BL}},\tan\beta_{\mathrm{TR}}\right)\le N_x/N_y\le\min\left(\tan\beta_{\mathrm{BR}},\tan\beta_{\mathrm{TL}}\right)\end{cases} \tag{37}$$

where equality means the boundary of loss of contact.

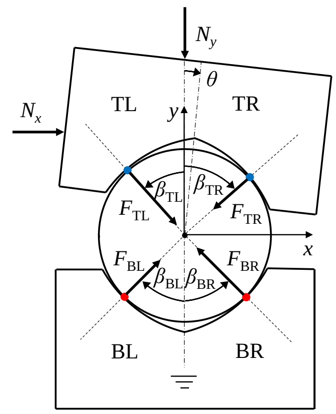

Figure 6. Contact force model

Define $\rho=N_x/N_y$ as the side force ratio and substitute Eq. (36) into the characteristic parameter K expressed in Eq. (23), they become

$$\begin{aligned}K_{\mathrm{BL}}&=\frac{N_y\left(\sin\beta_{\mathrm{BR}}-N_{x/y}\cos\beta_{\mathrm{BR}}\right)/\sin\left(\beta_{\mathrm{BL}}+\beta_{\mathrm{BR}}\right)}{\sin\left(\beta_{\mathrm{BR}}+\beta_{\mathrm{TR}}\right)+\sin\left(\beta_{\mathrm{TR}}+\beta_{\mathrm{TL}}\right)+\sin\left(\beta_{\mathrm{TL}}-\beta_{\mathrm{BR}}\right)},\\K_{\mathrm{BR}}&=\frac{N_y\left(\sin\beta_{\mathrm{BL}}+N_{x/y}\cos\beta_{\mathrm{BL}}\right)/\sin\left(\beta_{\mathrm{BL}}+\beta_{\mathrm{BR}}\right)}{\sin\left(\beta_{\mathrm{BL}}+\beta_{\mathrm{TL}}\right)+\sin\left(\beta_{\mathrm{TR}}+\beta_{\mathrm{TL}}\right)+\sin\left(\beta_{\mathrm{TR}}-\beta_{\mathrm{BL}}\right)},\\K_{\mathrm{TR}}&=\frac{N_y\left(\sin\beta_{\mathrm{TL}}-N_{x/y}\cos\beta_{\mathrm{TL}}\right)/\sin\left(\beta_{\mathrm{TL}}+\beta_{\mathrm{TR}}\right)}{\sin\left(\beta_{\mathrm{BL}}+\beta_{\mathrm{BR}}\right)+\sin\left(\beta_{\mathrm{BL}}+\beta_{\mathrm{TL}}\right)+\sin\left(\beta_{\mathrm{BR}}-\beta_{\mathrm{TL}}\right)},\\K_{\mathrm{TL}}&=\frac{N_y\left(\sin\beta_{\mathrm{TR}}+N_{x/y}\cos\beta_{\mathrm{TR}}\right)/\sin\left(\beta_{\mathrm{TL}}+\beta_{\mathrm{TR}}\right)}{\sin\left(\beta_{\mathrm{BL}}+\beta_{\mathrm{BR}}\right)+\sin\left(\beta_{\mathrm{BR}}+\beta_{\mathrm{TR}}\right)+\sin\left(\beta_{\mathrm{BL}}-\beta_{\mathrm{TR}}\right)}\end{aligned} \tag{38}$$

3.2. Case Study 1: Effect of Angular Misalignment

Angular misalignment is common in rolling element machine components. In this case study, the effect of angular misalignment θ is examined with all nominal contact angles of 45°. According to the definition of β in the proposed model, $\beta_{BL}=\beta_{BR}=45°$, $\beta_{TR}=45°+\theta$, $\beta_{TL}=45°-\theta$. Angular misalignment θ introduces deviation of the contact points, so that pure rolling condition does not hold.

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The rolling/sliding states are determined based on the relative values of K in Eq. (38), and plotted as a function of both ρ and θ in Fig. 7. The shaded area indicates loss of contact of some contact points where the condition in Eq. (37) is violated. The dotted lines indicate the boundary of loss of contact. At θ=0, the ball moves with pure rolling. The four regions indicate one-point sliding where the sliding contact point is denoted. Two-point sliding can happen at the intersection of two regions: along the red line, $K_{BL}= K_{TL}$, so BL and TL can be sliding simultaneously. Similarly, TR and BR can both slide along the blue line. The function of the lines for two-point sliding is explicitly given as

$$\rho = -\frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} \tag{39}$$

The rolling/sliding states in Fig. 7 are symmetric about the origin because of the symmetry of four-point point with respect to angular misalignment θ and side load N_x as shown in Fig. 6.

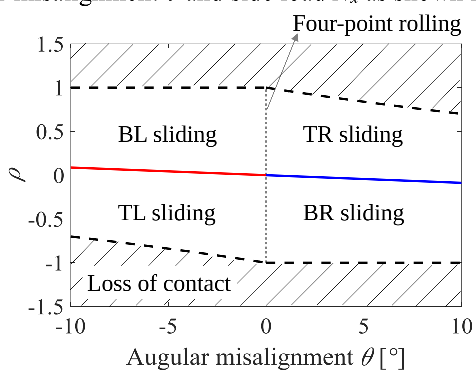

Figure 7. Sliding states as a function of angular misalignment and external loading

Friction loss in Eq. (24) can be formulated in this case study as

$$P_{f,\mathrm{X}} = K_\mathrm{X} 2v\mu_\mathrm{k} \left|\sin(\theta)\right|,\ \mathrm{X}\in\{\mathrm{BL},\mathrm{BR},\mathrm{TR},\mathrm{TL}\} \tag{40}$$

Without loss of generality, the unified friction loss is compared with the same v, μ_S and a fixed N_y as shown in Fig. 8. Generally, friction loss increases with increasing angular misalignment θ given the same loading condition. In application, angular misalignment should be avoided to reduce friction loss, which is common practice. A more interesting observation from Fig. 8 is: with the same angular misalignment, smaller side load (i.e., ρ=N_x/N_y) generally gives rise to more friction loss. The reason is that with smaller side load, the four contact points tend to have more evenly distributed contact forces. While with larger side load, some contact points have smaller contact forces (also smaller K), so sliding will take place at these locations. Because friction loss is proportional to K at the sliding contact point(s) as in Eq. (40), less friction loss comes as a result.

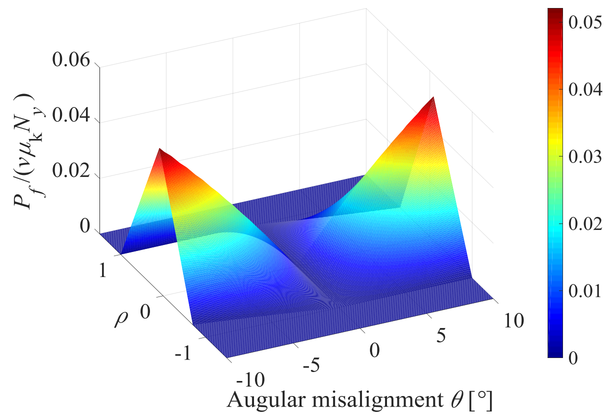

Figure 8. Relative friction loss with angular misalignment

Ball motion also depends on the rolling/sliding states of the ball. One typical example is shown here: when ρ=0 (i.e., N_x=0, no side load) and θ≥0, according to Fig. 7, the TR contact point will be sliding. Ball motion of TR sliding given in Eq. (11) becomes

$$\begin{Bmatrix} v_\mathrm{B} \\ \omega_x \\ \omega_y \end{Bmatrix} = \frac{v}{1+\cos\theta+\sin\theta}\begin{Bmatrix} 1 \\ \sqrt{2}/R_\mathrm{B} \\ 0 \end{Bmatrix} \tag{41}$$

Ball motion for θ=0 and θ>0 is visualized in Fig. 9. θ=0 sets the benchmark as pure rolling, where the ball translates at v/2 and rotates about x-axis. When θ>0, since the TR contact point is sliding, the zero-velocity lines pass through BL, BR and TL contact points as shown in Fig. 9(b). Although the ball still rotates about x-axis, both the translational and rotational velocities decrease with small θ.

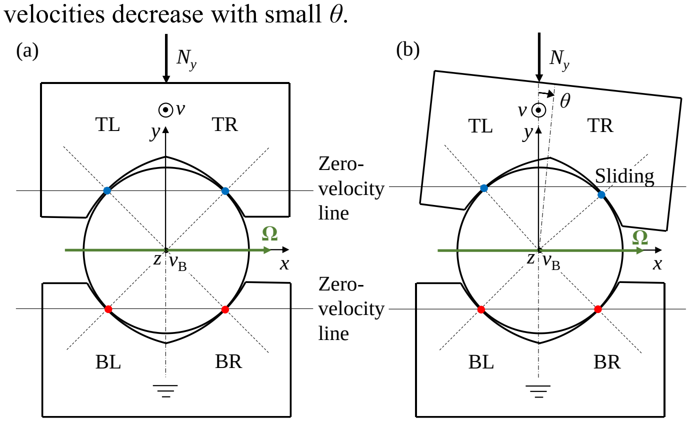

Figure 9. Ball motion with N_x=0 for: (a) θ=0; (b) θ>0

3.3. Case Study 2: Effect of Manufacturing Error

Manufacturing error is another common type of error in rolling element machine components. Consider the situation where all the contact points are supposed to have nominal contact angles of 45°. However, manufacturing error in TR groove makes its contact angle deviated with amount α as shown in Fig. 10. Thus β_{BL}=β_{BR}=β_{TL}=45°, β_{TR}=45°+α. Again pure rolling condition does not hold in this case.

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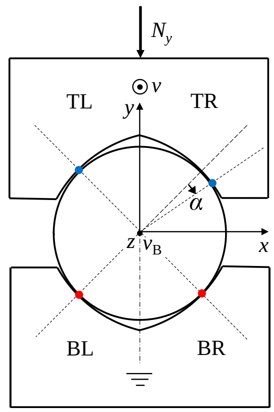

Figure 10. Contact angle deviation of the TR contact point

The rolling/sliding states as a function of both ρ and α is plotted in Fig. 11. The first major difference from Fig. 7 is that the sliding regions are no longer symmetric about the origin because positive and negative α introduce asymmetry to four-point contact. The other major difference is two-point sliding (BL and TR) exists because $K_{BL}=K_{TR}<K_{BR},K_{TL}$ for a wide range of positive ρ. Three-point sliding happens at the intersection of the two regions indicated by the red line and blue line.

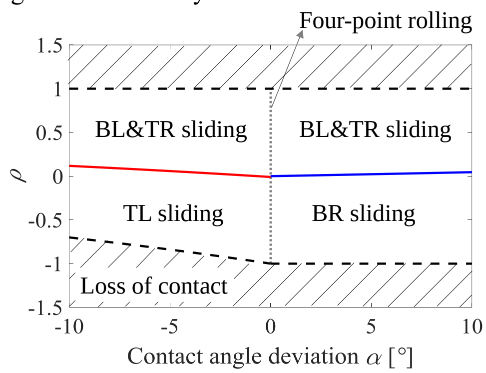

Figure 11. Sliding states as a function of TR contact angle deviation and external loading

The relative friction loss as a function of ρ and α is shown in Fig. 12. Generally, friction loss increases with increasing contact angle deviation α. With the same contact angle deviation, smaller side load (i.e., $\rho=N_x/N_y$) generally gives rise to more friction loss for the same reason in Case Study 1.

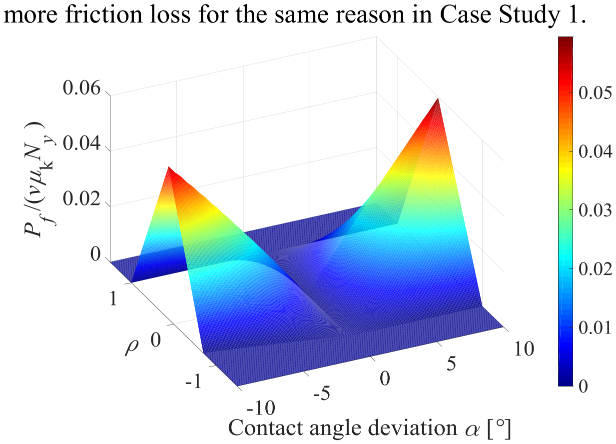

Figure 12. Relative friction loss with TR contact angle deviation

4. CONCLUSION AND FUTURE WORK

In this paper, a simplified analytical model for rolling/sliding behavior and friction in four-point contact has been developed based on Coulomb friction model and rigid body assumption. The internal over-constraint effect of four-point contact is illustrated: in order for pure rolling which has minimal friction loss to exist, the geometry of the four points need to satisfy certain relationship. Namely, the two lines passing through the two contact points on the same groove need to be parallel. If the pure rolling condition fails to hold, a combination of sliding and no sliding will happen. A characteristic parameter K for each contact point which incorporates normal contact forces and contact angles is derived. Sliding is proved to happen only at the contact point(s) with the smallest K. Based on the proposed model, analysis of the contributing factors to friction loss such as manufacturing errors, misalignments and external loading is investigated in case studies. Future work will compare the results of the proposed analytical model to those of comprehensive numerical models to assess its relative accuracy and usefulness for the design and analysis of four-point contact machine elements.

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ANNEX

A. 1. PROOF OF RANK(A_v)=RANK($\overline{A}_v$)=3

A_v in Eq. (7) is a 4×3 matrix, so rank(A_v)≤3. Pick three rows of A_v and form a new matrix $\overline{A}_v$ as presented in Eq. (12).

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$$\det(\bar{A}_v)=-\left(\sin\left(\beta_{\mathrm{BL}}+\beta_{\mathrm{BR}}\right)+\sin\left(\beta_{\mathrm{BL}}+\beta_{\mathrm{TL}}\right)+\sin\left(\beta_{\mathrm{BR}}-\beta_{\mathrm{TL}}\right)\right) \quad \text{(A1)}$$

From Eq. (A1),

$$\begin{aligned}&\sin\left(\beta_{\mathrm{BL}}+\beta_{\mathrm{BR}}\right)+\sin\left(\beta_{\mathrm{BL}}+\beta_{\mathrm{TL}}\right)+\sin\left(\beta_{\mathrm{BR}}-\beta_{\mathrm{TL}}\right)\\ &=\sin\left(\beta_{\mathrm{BL}}+\beta_{\mathrm{BR}}\right)\left(1+\cos\left(\beta_{\mathrm{BL}}+\beta_{\mathrm{TL}}\right)\right)\\ &\quad+\sin\left(\beta_{\mathrm{BL}}+\beta_{\mathrm{TL}}\right)\left(1-\cos\left(\beta_{\mathrm{BL}}+\beta_{\mathrm{TL}}\right)\right)\end{aligned} \quad \text{(A2)}$$

Since β_{BL}, β_{BR}, β_{TR} and $\beta_{\mathrm{TL}}\in(0, \pi/2)$, Eq. (A2)>0. So $\det(\bar{A}_v)<0$. Thus $\mathrm{rank}(A_v)= \mathrm{rank}(\bar{A}_v)=3$.

The term in Eq. (A2) is also the denominator of K_{TR} in Eq. (23). So K_{TR} is always positive. Same conclusion can be drawn that all the K values in Eq. (23) are positive following the similar procedure.

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