

Control-Oriented Modeling for Flexible Aircraft

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Flexible aircraft designs with high-aspect-ratio wings and lightweight structures are increasingly adopted to achieve improved fuel efficiency. To design a controller for a flexible aircraft, a low-order model of its dynamics that captures the aeroelastic behavior is needed. One approach to establishing such a low-order model is to extend the six-degree-of-freedom rigid aircraft model with additional elastic states. However, most of the existing models along these lines are based on simplified aerodynamics models (e.g., experimentally corrected aerodynamic coefficients), which limits the applicability of such low-order models in early design stages. In this work, a semi-analytical model of flexible aircraft is established through the combination of the dynamics of elastic body and the numerical linearization of high-order structure and aerodynamics models. The proposed model is derived in the body-fixed axes formulation, represents full flight dynamics (both longitudinal and lateral), and preserves the physical meanings for the states and all the derived terms. Static residualization is also adopted to enhance the model's accuracy with a limited number of elastic states. Using the proposed approach, low-order semi-analytical models are exemplified for different complexity aircraft representations and verified against the corresponding nonlinear high-order models.

Nomenclature

$C_{\beta T}, C_{\eta T}$	=	matrices linking the vector of thrust force F_T to $\{F_{\text{thrust}}^T, M_{\text{thrust}}^T\}^T$, and Q_{thrust}
$C_{\beta 0}, C_{\eta 0}$	=	aerodynamic forces on the rigid-body and flexible states at a selected trim condition, normalized by $\bar{q}$
$C_{\beta \beta}, C_{\beta \eta}, C_{\beta \dot{\eta}}, C_{\beta \delta}$	=	linearization coefficients of $\{F_{\text{aero}}^T, M_{\text{aero}}^T\}^T$, normalized by $\bar{q}$
$\bar{C}_{\beta \beta}, \bar{C}_{\beta \eta}, \bar{C}_{\beta \dot{\eta}}, \bar{C}_{\beta \delta}$	=	low-dimension modifications of $C_{\beta \beta}, C_{\beta \eta}, C_{\beta \dot{\eta}}, C_{\beta \delta}$ considering residualization
$C_{\eta \beta}, C_{\eta \eta}, C_{\eta \dot{\eta}}, C_{\eta \delta}$	=	linearization coefficients of Q_{aero} , normalized by $\bar{q}$
$\bar{C}_{\eta \beta}, \bar{C}_{\eta \eta}, \bar{C}_{\eta \dot{\eta}}, \bar{C}_{\eta \delta}$	=	low-dimension modifications of $C_{\eta \beta}, C_{\eta \eta}, C_{\eta \dot{\eta}}, C_{\eta \delta}$ considering residualization
D_k	=	matrix linking change of η to e_k
$E(x)$	=	complete nonlinear dynamics satisfying $E(x)\dot{x} = f_E(x) + g_E(x)u$
$f_E(x), g_E(x)$	=	vector of k th thrust force direction with and without aircraft deformation
$e_k, e_{0,k}$	=	
$\mathcal{F}_B, \mathcal{F}_I$	=	body-fixed and inertial reference frames
F_{Frame}	=	translational force components due to $\mathcal{F}_B$ movement (similar definitions for $F_{\text{Relative}}, F_{\text{Euler}}, F_{\text{Coriolis}}, F_{\text{Centripetal}}, F_{\text{Ext}}, F_g, F_{\text{thrust}}, F_{\text{aero}}, N$
$F_T, F_{T,k}$	=	vector of scalar thrust forces and its k th element, N

g, g_h	=	gravitation vector in $\mathcal{F}_B$ and gravitational constant as a function of altitude, $\text{m} \cdot \text{s}^{-2}$
$H(\eta), H_i(\eta)$	=	matrix linking $\ddot{\eta}$ to M_{Relative} and its i th column, $\text{N} \cdot \text{m} \cdot \text{s}^2$
$H_i^{(0)}, \nu_{ij}$	=	zeroth-order and first-order coefficients of $H_i(\eta)$, $\text{N} \cdot \text{m} \cdot \text{s}^2$
$I_B(\eta), I_B^{(0)}$	=	inertia matrix of deformed and undeformed aircraft, $\text{kg} \cdot \text{m}^2$
$J(\eta, \dot{\eta})$	=	matrix linking 2ω to M_{Coriolis} , $\text{N} \cdot \text{m} \cdot \text{s} \cdot \text{rad}^{-1}$
M, K	=	modal mass and stiffness matrix
M_{Frame}	=	rotational moment components due to $\mathcal{F}_B$ movement (similar definitions for $M_{\text{Relative}}, M_{\text{Euler}}, M_{\text{Coriolis}}, M_{\text{Centripetal}}, M_{\text{Ext}}, M_g, M_{\text{thrust}}, M_{\text{aero}}$), $\text{N} \cdot \text{m}$
M_{β}, M_{η}	=	aerodynamics coefficient modification matrices due to residualization
m	=	mass of the aircraft, kg
n, n_T	=	numbers of elastic modes and the thrust forces (engines)
O_B, O_I	=	origins of $\mathcal{F}_B$ and $\mathcal{F}_I$
p, q, r	=	roll, pitch, and yaw rates in $\mathcal{F}_B$, rad/s
Q_{Frame}	=	generalized force for flexible states due to $\mathcal{F}_B$ movement (similar for $Q_{\text{Relative}}, Q_{\text{Euler}}, Q_{\text{Elastic}}, Q_{\text{Coriolis}}, Q_{\text{Centripetal}}, Q_{\text{Ext}}, Q_g, Q_{\text{thrust}}, Q_{\text{aero}}$)
$\bar{q}, \bar{q}_t$	=	dynamic pressure and its trimmed value where aerodynamic linearization is acquired, Pa
$R(\phi, \theta)$	=	transformation matrix between the rates of Euler angles and ω
$r_{B/I}, r_{P/B}, r_{P/I}$	=	vectors from point I to B , B to P , and I to P , m
r_C, r_{C_0}	=	positions of center of mass of the deformed and undeformed aircraft in $\mathcal{F}_B$, m
$r_{E,k}, r_{E0,k}$	=	displacement of k th engine with and without aircraft deformation, m
r_P, r_{P_0}, e_{P_0}	=	deformed, undeformed position of Point P and their difference in $\mathcal{F}_B$, m
T	=	rotation matrix between $\mathcal{F}_B$ and $\mathcal{F}_I$
u	=	vector of combined control inputs $\{F_T^T, \delta^T\}^T$, N and rad
$v, {}^I v, {}^B v(v)$	=	aircraft velocity vector and its projection in $\mathcal{F}_I$ and $\mathcal{F}_B$, m/s

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v_x, v_y, v_z	= forward, lateral, and vertical speed in $\mathcal{F}_B$, m/s
x, x_t	= state vector including rigid-body and flexible states and its trimmed value
$\beta, \beta_t, \Delta\beta$	= combination of the translational and rotational rigid-body states, the trimmed value where aerodynamic linearization is acquired, and their difference, m/s and rad/s
Γ_i, Π_{ij}	= first- and second-order terms of $I_B(\eta)$ with respect to η , $\text{kg} \cdot \text{m}^2$
$\delta, \delta_t, \Delta\delta$	= control surface deflections, its trimmed value, and the deviation from the trim condition, rad
η, η_i	= vector of elastic states and the i th element
η_s, η_d	= vectors of the selected and discarded elements of η
$\eta_t, \Delta\eta$	= trimmed η and the deviation from the trim
Λ, Λ_i	= matrix linking η to the change of center of mass and its i th column, m
ρ	= altitude dependent air density, kg/m^3
ϕ, θ, ψ	= Euler angles, rad
ψ_i	= i th mode shape, m
Ω_0, Ω	= sets of undeformed and deformed points
ω, ω	= angular velocity vector and its projection in $\mathcal{F}_B$, rad/s

Subscripts

s	= selected modes in the residualization process
d	= discarded modes in the residualization process

I. Introduction

IN THE pursuit of cost and emission reduction, modern aircraft increasingly adopt high-aspect-ratio wings and lightweight structures, yielding a more flexible aircraft (FA) design. The FA design enhances structural and aerodynamics efficiency but poses challenges for modeling and control. Conventional controller designs use a six-degree-of-freedom (6-DOF) rigid aircraft model corrected by the aerodynamic flex-to-rigid ratio and with filters to suppress the elastic modes. In FA, a more direct handling of flexible modes must be taken into account; not accounting for flexible modes often leads to low-performance controllers [1] and more severe safety issues (e.g., flutter instability) [2,3]. On the other hand, high-order models of FA [4,5], which combine the finite element models, aerodynamic models, and rigid-body dynamics, typically have hundreds or thousands of states, and it is challenging to use them for controller design [6]. The use of high-order models in controller design may also lead to high-order control systems that require unrealistically high-performance or high-bandwidth actuators [7]. Therefore, to facilitate controller design, low-order FA models that capture the coupling of structural dynamics, aerodynamics, and rigid-body dynamics are needed.

The low-order models are typically obtained using one of two approaches. The top-to-bottom approach involves the use of model order reduction methods (e.g., balanced truncation [6,8], genetic algorithms [9], and Krylov-based reduction [10]). This numerical approach allows for great control over the level of approximation and the size of the final model. The top-to-bottom approaches can deal with significant complex dynamic models and have been successfully implemented in industry to model aeroelastic systems [11,12]. However, the physical meanings of the states and physics-based model structure may be lost in the model reduction process [13]. Furthermore, the available procedures are effective for developing linear parameter-varying models but are less effective for the development of models that account for and include nonlinearities and reflect the effects of physical parameter variations [14]. These potential issues may complicate the controller design and tuning process

(matching/interpolating models across different regions, observer design, etc.) of physical parameter variations and suffer from scalability challenges [14]. On the other hand, the bottom-to-top approach adds representations for the effects of flexibility on top of the analytical 6-DOF rigid aircraft model and thus preserves the physical meaning of the states and physics-based model structure. It yields an analytic or semi-analytical model, which can be conveniently reduced to a classical rigid aircraft dynamics model. The bottom-to-top approach is the focus of the present paper, with the motivation to provide low-order models for controller and observer designs.

The existing methods to incrementally integrate the FA dynamics into the model are categorized based on the reference frames adopted. Early investigations of Milne [15], Canavin and Linkins [16] exploited the concept of mean axes, which defines an orthogonal axis system with the origin at the instantaneous center of mass of the deformed body. The mean axes are oriented in such a way that the free vibration modes are decoupled from the rigid-body translational and rotational motion, and thus inherited the structure of rigid aircraft models in an elegant way [2,17]. Schmidt [18] proposed methods to characterize the elastic effects on aerodynamics forces and moments using analytical potential-flow aerodynamics coefficients. As Meirovitch and Tuzcu [19] pointed out, although the use of mean axes decoupled the inertial dynamics, the coupling effect is passed to the aerodynamic loads, which may complicate the computations. Accordingly, another commonly adopted reference frame is a body-fixed frame that has its origin at a fixed point in the aircraft. In this setting, the instantaneous center of mass is not at a constant location, and the inertia tensor varies. Although the equations of motion (EOM) are not inertially decoupled, the use of body-fixed axes permits to include the effect of the elastic deformations on the first- and second-mass moments of inertia, thereby potentially producing more accurate results [20]. Along these lines, Meirovitch and Tuzcu [21] followed a Lagrangian formulation that used a body-fixed reference frame (referred to as a “pseudo-body axes” frame) and incorporated the aerodynamics from strip theory. D’Eleuterio and Barfoot [22] used a similar energy-based approach that is applicable to general elastic bodies. Gibson et al. [23] proposed an FA model where the dihedral angles are used to represent the flexible states. Avanzini et al. proposed a mixed Newtonian–Lagrangian approach [7] to derive the equations of motion and employed the methodology to develop a simplified FA dynamics model [24]. Building on these existing works, researchers have also identified scenarios where the structure deformation shows significant geometrical nonlinearity [5]. Aircraft with these geometrical nonlinear deformations are referred to as very flexible aircraft (VFA). Compared to FA, VFA typically has large deformation amplitudes, and the geometrical nonlinearity hinders the linear structural analysis [25]. VFA modeling and control are significantly different from FA; this work focuses on the control-oriented modeling of FA.

The existing methods typically rely on simplified aerodynamics models (e.g., strip theory), which may not be satisfactory for accuracy and performance [19]. This paper builds on the bottom-to-top approach in the body-fixed axis setting and establishes a semi-analytical FA model exploiting the structural and aerodynamic linearizations of high-order models, such as in NASTRAN or in the University of Michigan’s Nonlinear Aeroelastic Simulation Toolbox (UM/NAST). The main contributions of the paper include deriving the full-flight semi-analytical FA model, developing an approach to extract the aerodynamics from high-order models, and processing linearization modifications to enhance accuracy and numerical stability. The proposed semi-analytical FA model is verified with a general transport aircraft (GTA) [26] and an Airbus XRF1 FA.

The paper is organized as follows: The body-fixed equations of motion of FA in the body-fixed frame are derived in Sec. II. The extraction of structural and aerodynamics information from high-order models, as well as the modifications to enhance accuracy and numerical stability, are discussed in Sec. III. The simulation case studies of the FA GTA and XRF1 models are detailed in Sec. IV. The conclusion and future work are discussed in Sec. V.

II. Semi-Analytical Low-Order Model of Flexible Aircraft

Consider an FA with mass m , an inertial frame $\mathcal{F}_I$ with origin O_I , and a body-fixed frame $\mathcal{F}_B$ with origin O_B (arbitrary point, but usually set as the aircraft undeformed center of mass), as illustrated in Fig. 1. Note that the deformation of the FA is exaggerated to highlight the deformation (such large deformations are typically categorized as VFA). The frame $\mathcal{F}_B$ is chosen in agreement with the north-east-down (NED) flight mechanics orientation and is related to frame $\mathcal{F}_I$ by a 3-2-1 Euler angle rotation sequence (defined by Euler angles ϕ , θ , and ψ). The rotation matrix T is defined as

$$T = \begin{bmatrix} c(\psi)c(\theta) & c(\psi)s(\phi)s(\theta) - c(\phi)s(\psi) & s(\phi)s(\psi) + c(\phi)c(\psi)s(\theta) \\ c(\theta)s(\psi) & c(\phi)c(\psi) + s(\phi)s(\psi)s(\theta) & c(\phi)s(\psi)s(\theta) - c(\psi)s(\phi) \\ -s(\theta) & c(\theta)s(\phi) & c(\phi)c(\theta) \end{bmatrix} \quad (1)$$

where $s(\cdot)$ and $c(\cdot)$ designate $\sin(\cdot)$ and $\cos(\cdot)$ functions. The rotation matrix equation (1) converts the coordinates in $\mathcal{F}_B$ to the ones in $\mathcal{F}_I$. For example, with the velocity of the aircraft is defined as $\mathbf{v} = ({}^I d/dt)\mathbf{r}_{B/I}$, its representations in $\mathcal{F}_I$ (i.e., ${}^I \mathbf{v}$) and in $\mathcal{F}_B$ (i.e., ${}^B \mathbf{v}$) satisfy the following relationship:

$$({}^I \mathbf{v}) = T({}^B \mathbf{v}) \quad (2)$$

In this work, a vector is projected onto the $\mathcal{F}_B$ frame without extra specification, e.g., $\mathbf{v} = {}^B \mathbf{v} = \{v_x, v_y, v_z\}^T$, where v_x , v_y , and v_z represent the forward, lateral, and vertical speed in $\mathcal{F}_B$ frame.

For an arbitrary point P on the aircraft, the vectors representing its position in reference frames $\mathcal{F}_I$ and $\mathcal{F}_B$ are written as $\mathbf{r}_{P/I}$ and $\mathbf{r}_{P/B}$, respectively. They satisfy the following relationship:

$$\mathbf{r}_{P/I} = \mathbf{r}_{P/B} + \mathbf{r}_{B/I} \quad (3)$$

where $\mathbf{r}_{B/I}$ is the inertial position of the vehicle (i.e., relative to the origin O_I of $\mathcal{F}_I$). Since the body-fixed frame is extensively used, a simplified convention is adopted for vectors that are expressed in $\mathcal{F}_B$ (i.e., $\mathbf{r}_P = \mathbf{r}_{P/B}$). The time derivatives with respect to reference frames $\mathcal{F}_I$ and $\mathcal{F}_B$ are denoted as ${}^I d/dt$ and ${}^B d/dt$, and they are linked through the transport theorem as

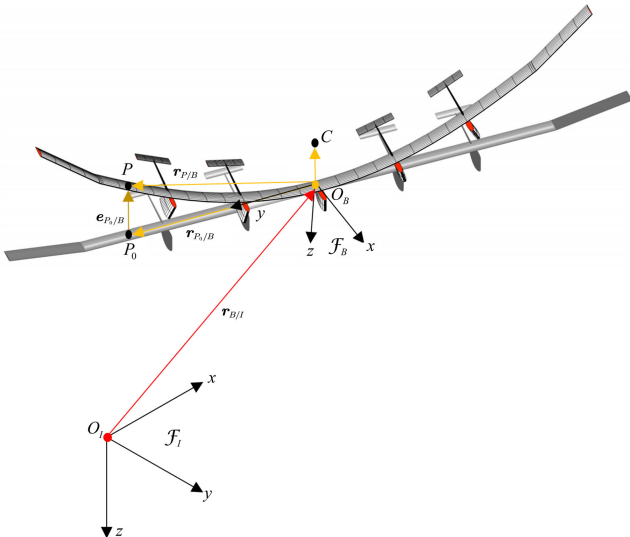


Fig. 1 Inertial and body-fixed reference frames. A typical very flexible aircraft configuration is used here just to simplify the visualization of the reference frames.

$$\frac{{}^I d}{dt} \mathbf{v} = \frac{{}^B d}{dt} \mathbf{v} + \boldsymbol{\omega} \times \mathbf{v} \quad (4)$$

where $\boldsymbol{\omega}$ is the angular velocity of $\mathcal{F}_B$ with respect to $\mathcal{F}_I$. We use $\dot{\mathbf{v}}$ and $\ddot{\mathbf{v}}$ as compact forms for $({}^B d/dt)\mathbf{v}$ and $({}^B d^2/dt^2)\mathbf{v}$. The roll, pitch, and yaw rates are defined as p , q , and r , i.e., $\boldsymbol{\omega} = \{p, q, r\}^T$. The cross product is written as a matrix multiplication by a skew-symmetric matrix (e.g., ${}^B(\boldsymbol{\omega} \times \mathbf{v}) \cong \tilde{\boldsymbol{\omega}}\mathbf{v}$). The skew-symmetric matrix $\tilde{\boldsymbol{\omega}}$ is defined as

$$\tilde{\boldsymbol{\omega}} = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \quad (5)$$

This definition of a skew-symmetric matrix applies to an arbitrary vector, and skew-symmetric matrices satisfy the properties $\tilde{\boldsymbol{\omega}}^T = -\tilde{\boldsymbol{\omega}}$ and $\tilde{\boldsymbol{\omega}}\mathbf{v} = -\tilde{\mathbf{v}}\boldsymbol{\omega}$. Note that the evolution of the Euler angles can be determined from

$$\frac{{}^I d}{dt} \begin{Bmatrix} \phi \\ \theta \\ \psi \end{Bmatrix} = \underbrace{\begin{bmatrix} 1 & s(\phi)t(\theta) & c(\phi)t(\theta) \\ 0 & c(\phi) & -s(\phi) \\ 0 & \frac{s(\phi)}{c(\theta)} & \frac{c(\phi)}{c(\theta)} \end{bmatrix}}_{R(\phi, \theta)} \begin{Bmatrix} p \\ q \\ r \end{Bmatrix} \quad (6)$$

where $R(\phi, \theta)$ designates the transformation matrix.

A. Equations of Motion

FA have moderate levels of flexibility, and their structural dynamics can be characterized using linear modal analysis. Unlike VFA, where the deformation displacement shows significant nonlinearity, in FA formulations, the position vector of an arbitrary point $\mathbf{r}_P$ is decomposed into the undeformed position $\mathbf{r}_{P_0}$ and the elastic deformation $\mathbf{e}_{P_0}$ as

$$\mathbf{r}_P = \mathbf{r}_{P_0} + \mathbf{e}_{P_0} = \mathbf{r}_{P_0} + \sum_{i=1}^n \psi_i(P_0)\eta_i \quad (7)$$

where $\psi_i(P_0)$ is the i th displacement mode shape at point P_0 , η_i is the modal amplitude of the i th mode, and n is the number of selected modes. Define Ω_0 and Ω to be the sets of undeformed and deformed points. Then the changes in the position of the center of mass due to deformation can be expressed as a linear function of the modal amplitudes, i.e.,

$$\mathbf{r}_C = \mathbf{r}_{C_0} + \sum_{i=1}^n \Lambda_i \eta_i = \mathbf{r}_{C_0} + \Lambda \boldsymbol{\eta}, \quad \Lambda_i = \frac{1}{m} \int_{P_0 \in \Omega_0} \psi_i(P_0) dm \quad (8)$$

where $\mathbf{r}_{C_0}$ is the position vector of the undeformed center of mass C_0 , Λ_i links the i th modal amplitude to the deformation of the center of mass, $\Lambda = [\Lambda_1, \Lambda_2, \dots, \Lambda_n]$, and $\boldsymbol{\eta} = \{\eta_1, \eta_2, \dots, \eta_n\}^T$.

The equation of motion for a point mass m_P in reference frame $\mathcal{F}_B$ is written as

$$m_P \left(\underbrace{\frac{{}^I d^2}{dt^2} \mathbf{r}_{B/I}}_{\text{Frame}} + \underbrace{\ddot{\mathbf{r}}_P}_{\text{Relative}} + \underbrace{\tilde{\boldsymbol{\omega}} \mathbf{r}_P}_{\text{Euler}} + \underbrace{2\tilde{\boldsymbol{\omega}} \dot{\mathbf{r}}_P}_{\text{Coriolis}} + \underbrace{\tilde{\boldsymbol{\omega}}^2 \mathbf{r}_P}_{\text{Centripetal}} \right) = \mathbf{F}_P \quad (9)$$

where $\mathbf{F}_P$ is the external force applied to the point mass. Note that acceleration terms are categorized into different classes: the inertial term due to the moving frame $\mathcal{F}_B$ is denoted as “Frame”; the inertial

term relative to frame $\mathcal{F}_B$ is denoted as “Relative”; the other terms are Euler, Coriolis, and centripetal terms. The integration of Eq. (9) with corresponding multipliers yields the translation, rotation, and elastic equations of motion for the FA. This classification of acceleration terms is crucial as it leads to all the following decomposition of translational, rotational, and flexible terms. The multipliers for translation, rotation, and elastic equations are 1, $\tilde{r}_P$, and $\psi_i^\top$, respectively.

The translation EOM is written as

$$F_{\text{Frame}} + F_{\text{Relative}} + F_{\text{Euler}} + F_{\text{Coriolis}} + F_{\text{Centripetal}} = F_{\text{Ext}} \quad (10)$$

where F_{Ext} represents the integral of the external forces applied to the FA, while F_{Frame} , F_{Relative} , F_{Euler} , F_{Coriolis} , and $F_{\text{Centripetal}}$ represent the integrals of corresponding terms in Eq. (9). These latter terms, by exploiting Eq. (8), are derived as

$$F_{\text{Frame}} = \int_{P \in \Omega} \left(\frac{d^2}{dt^2} r_{B/I} \right) dm = m \frac{d}{dt} v = m(\dot{v} + \tilde{\omega}v) \quad (11)$$

$$F_{\text{Relative}} = \int_{P \in \Omega} \ddot{r}_P dm = m\Lambda\ddot{\eta} \quad (12)$$

$$F_{\text{Euler}} = \int_{P \in \Omega} \tilde{\omega} r_P dm = -m\tilde{r}_C(\eta)\dot{\omega} \quad (13)$$

$$F_{\text{Coriolis}} = \int_{P \in \Omega} 2\tilde{\omega}\dot{r}_P dm = 2m\tilde{\omega}\Lambda\dot{\eta} \quad (14)$$

$$F_{\text{Centripetal}} = \int_{P \in \Omega} \tilde{\omega}^2 r_P dm = m\tilde{\omega}^2 r_C(\eta) \quad (15)$$

Although the corresponding force terms in Eqs. (10–15) are derived for FA (assuming linear structural modal decomposition), its formulations can be used to evaluate the VFA-related dynamics effects and potential error from the geometrical nonlinearity. For example, the center of mass $r_C(\eta)$ is an affine function of η for FA (assuming linear structural mode decomposition). Possible modeling errors may arise from this assumption of linear modes, particularly when FA increases flexibility toward VFA. To quantify this modeling error, an error function $e_{r_c}(\eta)$ is defined, marking the deviation of a nonlinear $r_C(\eta)$ (extracted from a high-order model) from the affine relationship defined in Eq. (8). This deviation is typically small with small η , and it gradually increases with larger η . Note from Eqs. (10–15) that this error function $e_{r_c}(\eta)$ directly links to the force estimation error that this model introduced due to the linear structural modes assumption. On the other hand, this error function $e_{r_c}(\eta)$ also helps establish the admissible domain of simulation (bounds on the flexible states η in simulation).

Similarly, integrating Eq. (9) with multiplier $\tilde{r}_P$ yields the rotational EOM

$$M_{\text{Frame}} + M_{\text{Relative}} + M_{\text{Euler}} + M_{\text{Coriolis}} + M_{\text{Centripetal}} = M_{\text{Ext}} \quad (16)$$

where M_{Ext} is the external moment applied to the FA using B as the reference point. The terms M_{Frame} and M_{Relative} are given by

$$M_{\text{Frame}} = \int_{P \in \Omega} \left(\tilde{r}_P \frac{d^2}{dt^2} r_{B/I} \right) dm = m\tilde{r}_C(\eta)(\dot{v} + \tilde{\omega}v) \quad (17)$$

$$M_{\text{Relative}} = \int_{P \in \Omega} (\tilde{r}_P \ddot{r}_P) dm = H(\eta)\ddot{\eta} \quad (18)$$

where $H(\eta) = [H_1(\eta), H_2(\eta), \dots, H_n(\eta)]$. Each term $H_i(\eta)$ ($i = 1, 2, \dots, n$) is affine in η and can be expressed in the form

$$H_i(\eta) = H_i^{(0)} + \nu_{ij}\eta_j \quad (19)$$

where

$$\begin{aligned} H_i^{(0)} &= \int_{P_0 \in \Omega_0} \tilde{r}_{P_0} \psi_i(P_0) dm, \\ \nu_{ij} &= \int_{P_0 \in \Omega_0} (\tilde{\psi}_j(P_0) \psi_i(P_0)) dm \end{aligned} \quad (20)$$

From Eq. (20), it follows that $\nu_{ij} = -\nu_{ji}$. The term M_{Euler} is given by

$$M_{\text{Euler}} = \int_{P \in \Omega} \tilde{r}_P \tilde{\omega} r_P dm = I_B(\eta)\dot{\omega} \quad (21)$$

where $I_B(\eta) = -\int_{P \in \Omega} \tilde{r}_P \tilde{r}_P dm$ is the inertia matrix, whose elements are second-order polynomials in η as

$$I_B(\eta) = I_B^{(0)} + (\Gamma_i + \Gamma_i^\top)\eta_i + \Pi_{ij}\eta_i\eta_j \quad (22)$$

The term $I_B^{(0)} = -\int_{P_0 \in \Omega_0} (\tilde{r}_{P_0} \tilde{r}_{P_0}) dm$ in Eq. (22) is the inertia matrix of the undeformed aircraft, and Γ_i and Π_{ij} are given by

$$\begin{aligned} \Gamma_i &= -\int_{P_0 \in \Omega_0} \tilde{r}_{P_0} \tilde{\psi}_i(P_0) dm, \\ \Pi_{ij} &= -\int_{P_0 \in \Omega_0} \tilde{\psi}_i(P_0) \tilde{\psi}_j(P_0) dm \end{aligned} \quad (23)$$

Due to the properties of the skew-symmetric matrices, it follows that $\Pi_{ij}^\top = \Pi_{ji}$. The term M_{Coriolis} is given by

$$M_{\text{Coriolis}} = \int_{P \in \Omega} 2\tilde{r}_P \tilde{\omega} \dot{r}_P dm = 2J(\eta, \dot{\eta})\omega \quad (24)$$

where

$$J(\eta, \dot{\eta}) = (\Gamma_i + \Pi_{ji}\eta_j)\dot{\eta}_i \quad (25)$$

The term $M_{\text{Centripetal}}$ is given by

$$M_{\text{Centripetal}} = \int_{P \in \Omega} \tilde{r}_P \tilde{\omega} \tilde{\omega} r_P dm = \tilde{\omega} I_B(\eta) \omega \quad (26)$$

where the right-hand side is obtained from the property of the cross product ($\tilde{u}_1 \tilde{u}_2 \tilde{u}_2 u_1 = -\tilde{u}_2 \tilde{u}_1 \tilde{u}_1 u_2$), which holds for arbitrary vectors u_1 and u_2 .

The flexible EOM is derived similarly, through the integration of Eq. (9) with multiplier $\psi_i^\top$. The dynamics of the i th mode is given by

$$\begin{aligned} Q_{\text{Frame},i} + Q_{\text{Relative},i} + Q_{\text{Elastic},i} + Q_{\text{Euler},i} + Q_{\text{Coriolis},i} \\ + Q_{\text{Centripetal},i} = Q_{\text{Ext},i} \end{aligned} \quad (27)$$

The terms $Q_{\text{Frame},i}$ and $Q_{\text{Relative},i}$ are given by

$$Q_{\text{Frame},i} = \int_{P \in \Omega} \psi_i^\top \left(\frac{d^2}{dt^2} r_{B/I} \right) dm = m\Lambda_i^\top (\dot{v} + \tilde{\omega}v) \quad (28)$$

$$Q_{\text{Relative},i} = \int_{P \in \Omega} \psi_i^\top \ddot{r}_P dm = M_{ij}\ddot{\eta}_j \quad (29)$$

where M_{ij} is the element of modal mass matrix M defined as

$$M_{ij} = \int_{P \in \Omega} \psi_i^\top \psi_j dm \quad (30)$$

Similarly, define K_{ij} as the element of the modal stiffness matrix K . Then, the elastic terms are given by

$$Q_{\text{Elastic},i} = K_{ij}\eta_j \quad (31)$$

Based on the definitions of $H_i(\eta)$, ν_{ij} , $\tilde{\Gamma}_i$, and $\tilde{\Pi}_{ij}$ in Eqs. (19), (20), and (23), it follows that $Q_{\text{Euler},i}$, $Q_{\text{Coriolis},i}$, and $Q_{\text{Centripetal},i}$ are given by

$$Q_{\text{Euler},i} = \int_{P \in \Omega} \psi_i^\top \tilde{\omega} r_P dm = H_i^\top(\eta) \dot{\omega} \quad (32)$$

$$Q_{\text{Coriolis},i} = \int_{P \in \Omega} 2\psi_i^\top \tilde{\omega} \dot{r}_P dm = 2\omega^\top \nu_{ij} \dot{\eta}_j \quad (33)$$

$$Q_{\text{Centripetal},i} = \int_{P \in \Omega} \psi_i^\top \tilde{\omega}^2 r_P dm = -\omega^\top (\Gamma_i^\top + \Pi_{ij} \eta_j) \omega \quad (34)$$

Similar modeling error nonlinear functions due to the linear assumption are also quantifiable following the error function establishment procedure of $e_{r_c}(\eta)$ for the rotational EOM and flexible EOM. More specifically, $H(\eta)$, $I_B(\eta)$, and $J(\eta, \dot{\eta})$ are the terms that can be extracted from high-order models. In comparison with the linear definitions, corresponding error functions $e_H(\eta)$, $e_{I_B}(\eta)$, and $e_J(\eta, \dot{\eta})$ can be established. Similar to the establishment of $e_{r_c}(\eta)$, the error due to the linear assumption or the bounds of the flexible states can be established with these error functions.

B. Forces and Moments

The external forces, moments, and generalized forces are categorized into gravity, thrust, and aerodynamics components:

$$\begin{aligned} F_{\text{Ext}} &= F_g + F_{\text{thrust}} + F_{\text{aero}}, \\ M_{\text{Ext}} &= M_g + M_{\text{thrust}} + M_{\text{aero}}, \\ Q_{\text{Ext},i} &= Q_{g,i} + Q_{\text{thrust},i} + Q_{\text{aero},i} \end{aligned} \quad (35)$$

1. Gravity Forces and Moments

The gravity-related terms F_g , M_g , and $Q_{g,i}$ are given by

$$\begin{aligned} F_g &= mg, \\ M_g &= \int_{P \in \Omega} (\tilde{r}_P g) dm = m \tilde{r}_C(\eta) g, \\ Q_{g,i} &= \int_{P \in \Omega} (\psi_i^\top g) dm = m \Lambda_i^\top g \end{aligned} \quad (36)$$

where g is the gravity constant vector pointing to the Earth. According to the transformation matrix defined in Eq. (1), g in $\mathcal{F}_B$ is given by $g = T^{-1}\{0, 0, -g_h\}^\top$, in which the gravity constant g_h is a function of the altitude.

2. Thrust Forces and Moments

Assume that unit vectors $e_{0,k}$ and e_k ($k = 1, 2, \dots, n_T$) represent the thrust force directions of n_T engines without and with deformation, respectively. Unlike in rigid aircraft where $e_{0,k} = e_k$, thrust force direction e_k in FA models is a function of η . For small deformations, the rotations due to different modes are additive so that

$$e_k = e_{0,k} + D_k \eta \quad (37)$$

where

$$D_k = [\tilde{n}_{k,1} e_{0,k}, \tilde{n}_{k,2} e_{0,k}, \dots, \tilde{n}_{k,n} e_{0,k}] \quad (38)$$

and where $n_{k,i}$ is the rotation vector at the k th engine location due to unit-amplitude i th mode. Define the thrust force of the k th engine as $F_{T,k}$. Then the total thrust force and moment are expressed as

$$F_{\text{thrust}} = \sum_{k=1}^{n_T} F_{T,k} e_k \quad (39)$$

$$M_{\text{thrust}} = \sum_{k=1}^{n_T} F_{T,k} \tilde{r}_{E,k} e_k \quad (40)$$

The vector $r_{E,k}$ represents the k th engine's displacement in the deformed configuration given by

$$r_{E,k} = r_{E0,k} + \sum_{i=1}^n \psi_i(E_k) \eta_i \quad (41)$$

where $\psi_i(E_k)$ is the i th mode shape at the k th engine location. The generalized thrust forces that appear in the dynamic equations for flexible states are calculated as

$$Q_{\text{thrust},i} = \sum_{k=1}^{n_T} F_{T,k} e_k \cdot \frac{\partial r_{E,k}}{\partial \eta_i} = \sum_{k=1}^{n_T} F_{T,k} e_k^\top \psi_i(E_k) \quad (42)$$

In vector format, $F_T = \{F_{T,1}, F_{T,2}, \dots, F_{T,n_T}\}^\top$ and the thrust force, moment, and generalized forces are given by

$$\begin{aligned} \begin{Bmatrix} F_{\text{thrust}} \\ M_{\text{thrust}} \end{Bmatrix} &= C_{\beta T} F_T, \\ Q_{\text{thrust}} &= C_{\eta T} F_T \end{aligned} \quad (43)$$

where

$$C_{\beta T} = \begin{bmatrix} e_1 & e_2 & \dots & e_{n_T} \\ \tilde{r}_{E,1} e_1 & \tilde{r}_{E,2} e_2 & \dots & \tilde{r}_{E,n_T} e_{n_T} \end{bmatrix} \quad (44)$$

$$C_{\eta T} = \begin{bmatrix} e_1^\top \psi_1(E_1) & e_2^\top \psi_1(E_2) & \dots & e_{n_T}^\top \psi_1(E_{n_T}) \\ \vdots & \vdots & & \vdots \\ e_1^\top \psi_n(E_1) & e_2^\top \psi_n(E_2) & \dots & e_{n_T}^\top \psi_n(E_{n_T}) \end{bmatrix} \quad (45)$$

3. Aerodynamics Forces and Moments

This semi-analytical FA model exploits the linearization of high-order models to obtain characterizations of aerodynamics forces and moments. Define the vector of the rigid-body states $\beta = \{v^\top, \omega^\top\}^\top$ as the combination of the translation and rotation velocity states and let δ aggregates the inputs from all control effectors.

The aerodynamics force and moment are then written as

$$\begin{aligned} \begin{Bmatrix} F_{\text{aero}} \\ M_{\text{aero}} \end{Bmatrix} &= \bar{q} (C_{\beta 0} + C_{\beta \beta} \Delta \beta + C_{\beta \eta} \Delta \eta + C_{\beta \dot{\eta}} \dot{\eta} + C_{\beta \delta} \Delta \delta), \\ Q_{\text{aero}} &= \bar{q} (C_{\eta 0} + C_{\eta \beta} \Delta \beta + C_{\eta \eta} \Delta \eta + C_{\eta \dot{\eta}} \dot{\eta} + C_{\eta \delta} \Delta \delta) \end{aligned} \quad (46)$$

where $C_{\beta 0}$ and $C_{\eta 0}$ represent the normalized aerodynamic forces affecting the rigid and flexible states at a selected trim condition, where the dynamic pressure $\bar{q} = (\rho \|v\|^2 / 2)$ (with ρ being the altitude-dependent air density) is introduced as the normalization factor. When perturbed from this selected trim condition, the aerodynamics are assumed to follow a linear relationship around the selected trim point, where

$$\begin{aligned} \Delta \beta &= \beta - \beta_t, \\ \Delta \eta &= \eta - \eta_t, \\ \Delta \delta &= \delta - \delta_t \end{aligned} \quad (47)$$

are the deviations from the corresponding trim condition values of β_t , η_t , and δ_t . As shown in Eq. (46), their sensitivity coefficients to the rigid states (i.e., $C_{\beta \beta}$, $C_{\beta \eta}$, and $C_{\beta \delta}$) and to the flexible states (i.e., $C_{\eta \beta}$, $C_{\eta \eta}$, and $C_{\eta \delta}$) are defined with $\bar{q}$ normalization as well. Note that more practical FA implementation may require multiple trim

points and more complex aerodynamics models. To account for these models, interpolation of the linear models or even nonlinear surrogate models can replace the simplified aerodynamics model in Eq. (46). Under those scenarios, the coefficients shown in Eq. (46) can be interpreted as the local linearizations of a more complex model to account for the aeroelastic analysis and the aerodynamic linearization matching method discussed in Sec. III.B.

C. Summary of the Equations of Motion

Let the state vector x aggregate the rigid-body states, flexible states, Euler angles, and the position of the body-fixed frame, i.e.,

$$x = \{\beta^\top, \eta^\top, \dot{\eta}^\top, \phi, \theta, \psi, r_{B/I}^\top\}^\top \quad (48)$$

and let the n_u -dimensional control input u aggregate the thrust forces and the control surface deflections, i.e.,

$$u = \{F_T^\top, \delta^\top\}^\top \quad (49)$$

Based on all the dynamics terms calculated in this section, the FA dynamics are represented as

$$E(x)\dot{x} = f_E(x) + g_E(x)u \quad (50)$$

where

$$E(x) = \begin{bmatrix} mI & -m\tilde{r}_C(\eta) & 0 & m\Lambda & 0 & 0 \\ m\tilde{r}_C(\eta) & I_B(\eta) & 0 & H(\eta) & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ m\Lambda^\top & H^\top(\eta) & 0 & M & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix} \quad (51)$$

$$f_E(x) = \begin{bmatrix} -m\tilde{\omega}v - 2m\tilde{\omega}\Lambda\dot{\eta} - m\tilde{\omega}^2 r_C(\eta) + mg + F_{\text{aero}}|_{\delta=0} \\ -m\tilde{r}_C(\eta)\tilde{\omega}v - 2J(\eta, \dot{\eta})\omega - \tilde{\omega}I_B(\eta)\omega + m\tilde{r}_C(\eta)g + M_{\text{aero}}|_{\delta=0} \\ \dot{\eta} \\ -K\eta - m\Lambda^\top \tilde{\omega}v - Q_{\text{Coriolis}} - Q_{\text{Centripetal}} + m\Lambda^\top g + Q_{\text{aero}}|_{\delta=0} \\ R(\phi, \theta)\omega \\ Tv \end{bmatrix} \quad (52)$$

and

$$g_E(x) = \begin{bmatrix} C_{\beta T} & \bar{q}C_{\beta\delta} \\ 0_{n \times n_u} \\ C_{\eta T} & \bar{q}C_{\eta\delta} \\ 0_{6 \times n_u} \end{bmatrix} \quad (53)$$

III. Structural and Aerodynamics Information from High-Order Models

It is assumed that the structural modes ψ_i ($i = 1, \dots, n$) are acquired from the high-order models such as NASTRAN and UM/NAST. Note that ψ_i is the displacement-based mode shape, and other type of mode shapes (e.g., strain-based mode shapes in UM/NAST) should be converted to displacement mode shape with linear transformation matrices. This linear transformation is possible due to the linear deformation assumption in FA [23]. The extraction of structural modes provides the mass matrix M , the stiffness matrix K , and mode shapes ψ_i , with which other variables Λ , $H_i^{(0)}$, Γ_i , ν_{ij} , and Π_{ij}

($i = 1, \dots, n, j = 1, \dots, n$) are calculated accordingly. The aerodynamics coefficients are calculated based on two different scenarios. The first scenario assumes that the direct linearizations of F_{aero} , M_{aero} , and Q_{aero} with respect to the rigid states, flexible states, and control surface deflections are available, while the second scenario assumes that a general state-space linear-time-invariant system realization is available. These two scenarios are detailed in the following subsections.

A. Model-Order Reduction with Static Residualization

Assume that the high-order aerodynamic solver provides a linearization of F_{aero} , M_{aero} , and Q_{aero} at a trimmed point x_t with dynamic pressure $\bar{q}_t$. The coefficients in Eq. (46) are given by

$$C_{\beta 0} = \frac{1}{\bar{q}_t} \left\{ \begin{matrix} F_{\text{aero},t} \\ M_{\text{aero},t} \end{matrix} \right\}, \quad C_{\beta\chi} = \frac{1}{\bar{q}_t} \frac{\partial}{\partial \chi} \left\{ \begin{matrix} F_{\text{aero}} \\ M_{\text{aero}} \end{matrix} \right\},$$

$$C_{\eta 0} = \frac{Q_{\text{aero},t}}{\bar{q}_t}, \quad C_{\eta\chi} = \frac{1}{\bar{q}_t} \frac{\partial Q_{\text{aero}}}{\partial \chi}, \quad (\chi \in \{\beta, \eta, \dot{\eta}, \delta\}) \quad (54)$$

There are typically thousands of flexible states in the high-order models. To facilitate reduced-order modeling, only a small portion of selected modes is included in our adopted aerodynamics model. Let η be decomposed according to the selected modes and the discarded modes, i.e.,

$$\eta = \begin{Bmatrix} \eta_s \\ \eta_d \end{Bmatrix} \quad (55)$$

and let other matrices be also decomposed accordingly:

$$C_{\beta\eta} = [C_{\beta\eta,s} \quad C_{\beta\eta,d}], \quad C_{\beta\dot{\eta}} = [C_{\beta\dot{\eta},s} \quad C_{\beta\dot{\eta},d}],$$

$$C_{\eta 0} = \begin{Bmatrix} C_{\eta 0,s} \\ C_{\eta 0,d} \end{Bmatrix}, \quad C_{\eta\eta} = \begin{bmatrix} C_{\eta\eta,ss} & C_{\eta\eta,sd} \\ C_{\eta\eta,ds} & C_{\eta\eta,dd} \end{bmatrix},$$

$$C_{\eta\dot{\eta}} = \begin{bmatrix} C_{\eta\dot{\eta},ss} & C_{\eta\dot{\eta},sd} \\ C_{\eta\dot{\eta},ds} & C_{\eta\dot{\eta},dd} \end{bmatrix}, \quad C_{\eta\delta} = \begin{bmatrix} C_{\eta\delta,s} \\ C_{\eta\delta,d} \end{bmatrix},$$

$$M = \begin{bmatrix} M_{ss} & M_{sd} \\ M_{ds} & M_{dd} \end{bmatrix}, \quad K = \begin{bmatrix} K_{ss} & K_{sd} \\ K_{ds} & K_{dd} \end{bmatrix} \quad (56)$$

If the η_d -related terms are simply discarded by setting all the matrix with subscript “d” to zero matrices, this could lead to significant model inaccuracy since the effects of discarded modes are not considered. As suggested in Ref. [27], steady-state responses of η_d can be included in the model to enhance the model accuracy, which is referred to as static residualization. To enable the residualization, the steady-state final value $\Delta\eta_d|_{t \rightarrow \infty}$ is calculated based on the following relationship:

$$\Delta\eta_d|_{t \rightarrow \infty} = [0 \quad I](K - \bar{q}_t C_{\eta\eta})^{-1} \bar{q}_t (C_{\eta\dot{\eta}} \Delta\dot{\eta}|_{t \rightarrow \infty} + C_{\eta\delta} \Delta\delta|_{t \rightarrow \infty} + C_{\eta\beta} \Delta\beta|_{t \rightarrow \infty}) \quad (57)$$

The residualization considers the effect of $\Delta\eta_d|_{t \rightarrow \infty}$ through the modifications of the coefficients in the selected modes as

$$\bar{C}_{\beta\eta} = C_{\beta\eta,s},$$

$$\bar{C}_{\beta\dot{\eta}} = C_{\beta\dot{\eta},s} + M_{\beta} C_{\eta\dot{\eta}},$$

$$\bar{C}_{\beta\beta} = C_{\beta\beta,s} + M_{\beta} C_{\eta\beta},$$

$$\bar{C}_{\beta\delta} = C_{\beta\delta,s} + M_{\beta} C_{\eta\delta},$$

$$\bar{C}_{\eta\eta} = C_{\eta\eta,ss},$$

$$\bar{C}_{\eta\dot{\eta}} = C_{\eta\dot{\eta},ss} - M_{\eta} C_{\eta\dot{\eta}},$$

$$\bar{C}_{\eta\beta} = C_{\eta\beta,s} - M_{\eta} C_{\eta\beta},$$

$$\bar{C}_{\eta\delta} = C_{\eta\delta,s} - M_{\eta} C_{\eta\delta} \quad (58)$$

where the modification matrices are given by

$$\begin{aligned} M_\beta &= C_{\beta\eta,d}[0 \quad I](K - \bar{q}_t C_{\eta\eta})^{-1}, \\ M_\eta &= (K_{sd} - \bar{q}_t C_{\eta\eta,sd})[0 \quad I](K - \bar{q}_t C_{\eta\eta})^{-1} \end{aligned} \quad (59)$$

Low-order models of the FA are established with these reduced-order coefficients matrices.

B. Aerodynamic Coefficients from Linearization Matching

Certain high-fidelity modeling tools (e.g., UM/NAST) provide linearized models in the state-space form. Therefore, the aerodynamic coefficients can also be acquired by matching the linearization at the trim point. This approach is similar to the modal matching concept in [28]. Define the trimmed velocity, angular rate, center-of-mass location, and elastic states to be v_t , ω_t , $r_{C,t}$, and η_t , respectively. Then the linearization of the translation terms is given in Table 1. Define

$$\Gamma_\omega = [\bar{\Gamma}_1 \omega_t, \bar{\Gamma}_2 \omega_t, \dots, \bar{\Gamma}_n \omega_t] \quad (60)$$

where $\bar{\Gamma}_i = \Gamma_i + \Gamma_i^\top$ ($i = 1, 2, \dots, n$) and

$$J_\omega = [(\Gamma_1 + \Pi_{j1} \eta_{j,t}) \omega_t, (\Gamma_2 + \Pi_{j2} \eta_{j,t}) \omega_t, \dots, (\Gamma_n + \Pi_{jn} \eta_{j,t}) \omega_t] \quad (61)$$

The rotational terms and their linearization are given in Table 2. Similarly, define matrices ν_ω and Π_ω with their individual terms given by

$$\nu_{\omega,ij} = \omega_t^\top \nu_{ij} \quad (62)$$

$$\Pi_{\omega,ij} = \omega_t^\top \Pi_{ij} \omega_t \quad (63)$$

The flexible terms and their linearization are given in Table 3.

$$A_{r0} = \begin{bmatrix} -m\tilde{\omega}_t & m(\tilde{v}_t - 2\tilde{\omega}_t \tilde{r}_{C,t} - \tilde{r}_{C,t} \tilde{\omega}_t) & -m\tilde{\omega}_t^2 \Lambda & -2m\tilde{\omega}_t \Lambda \\ -m\tilde{r}_{C,t} \tilde{\omega}_t & m\tilde{r}_{C,t} \tilde{v}_t - (\tilde{\omega}_t I_{B,t} - I_{B,t} \tilde{\omega}_t) & m(\tilde{\omega}_t \tilde{v}_t - \tilde{v}_t \tilde{\omega}_t) \Lambda - \Gamma_\omega + \tilde{\omega}_t \Gamma_\omega & -2J_\omega \\ 0 & 0 & 0 & I \\ -m\Lambda^\top \tilde{\omega}_t & m\Lambda^\top \tilde{v}_t & \Pi_\omega - K & -2\nu_\omega \end{bmatrix} \quad (67)$$

Table 1 Translation terms and linearization

Term	Expression	Linearization term	Expression
F_{Frame}	$m(\dot{v} + \tilde{\omega}v)$	ΔF_{Frame}	$m\Delta\dot{v} + m\tilde{\omega}_t \Delta v - m\tilde{v}_t \Delta\omega$
F_{Inertial}	$m\Lambda\ddot{\eta}$	$\Delta F_{\text{Inertial}}$	$m\Lambda\Delta\ddot{\eta}$
F_{Euler}	$-m\tilde{r}_C(\eta)\dot{\omega}$	ΔF_{Euler}	$-m\tilde{r}_{C,t} \Delta\dot{\omega}$
F_{Coriolis}	$2m\tilde{\omega}\Lambda\dot{\eta}$	$\Delta F_{\text{Coriolis}}$	$2m\tilde{\omega}_t \Lambda\Delta\dot{\eta}$
$F_{\text{Centrifugal}}$	$m\tilde{\omega}^2 r_C(\eta)$	$\Delta F_{\text{Centrifugal}}$	$m(2\tilde{\omega}_t \tilde{r}_{C,t} + \tilde{r}_{C,t} \tilde{\omega}_t) \Delta\omega + m\tilde{\omega}_t^2 \Lambda\Delta\eta$

Table 2 Rotational terms and linearization

Term	Expression	Linearization term	Expression
M_{Frame}	$m\tilde{r}_C(\eta)(\dot{v} + \tilde{\omega}v)$	ΔM_{Frame}	$m\tilde{r}_{C,t} \Delta\dot{v} + m\tilde{r}_{C,t} \tilde{\omega}_t \Delta v - m(\tilde{\omega}_t \tilde{v}_t - \tilde{v}_t \tilde{\omega}_t) \Lambda \Delta\eta - m\tilde{r}_{C,t} \tilde{v}_t \Delta\omega$
M_{Inertial}	$H(\eta)\ddot{\eta}$	$\Delta M_{\text{Inertial}}$	$H(\eta_t) \Delta\ddot{\eta}$
M_{Euler}	$I_B(\eta)\dot{\omega}$	ΔM_{Euler}	$I_{B,t} \Delta\dot{\omega} + \Gamma_\omega \Delta\eta$
M_{Coriolis}	$2J(\eta, \dot{\eta})\omega$	$\Delta M_{\text{Coriolis}}$	$2J_\omega \Delta\dot{\eta}$
$M_{\text{Centrifugal}}$	$\tilde{\omega} I_B(\eta)\omega$	$\Delta M_{\text{Centrifugal}}$	$(\tilde{\omega}_t I_{B,t} - I_{B,t} \tilde{\omega}_t) \Delta\omega + \tilde{\omega}_t \Gamma_\omega \Delta\eta$

Table 3 Flexible terms and linearization

Term	Expression	Linearization term	Expression
$Q_{\text{Frame},i}$	$m\Lambda_i^\top (\dot{v} + \tilde{\omega}v)$	ΔQ_{Frame}	$m\Lambda^\top (\Delta\dot{v} + \tilde{\omega}_t \Delta v - \tilde{v}_t \Delta\omega)$
$Q_{\text{Inertial},i} + Q_{\text{Elastic},i}$	$M_{ij}\ddot{\eta}_j + K_{ij}\eta_j$	$\Delta Q_{\text{Inertial}}$	$M\Delta\ddot{\eta} + K\Delta\eta$
$Q_{\text{Euler},i}$	$H_i^\top(\eta)\dot{\omega}$	ΔQ_{Euler}	$H^\top(\eta_t) \Delta\dot{\omega}$
$Q_{\text{Coriolis},i}$	$2\omega^\top \nu_{ij} \dot{\eta}_j$	$\Delta Q_{\text{Coriolis}}$	$2\nu_\omega \Delta\dot{\eta}$
$Q_{\text{Centrifugal},i}$	$-\omega^\top (\Gamma_i^\top + \Pi_{ij}\eta_j)\omega$	$\Delta Q_{\text{Centrifugal}}$	$-\Pi_\omega \Delta\eta$

Assuming small changes in Euler angles and altitudes, the state-space linearization of the system at the trim condition is given by

$$E_r \Delta \dot{x}_r = A_r \Delta x_r + B_r \Delta u \quad (64)$$

where $\Delta x_r = \{\Delta v^\top, \Delta \omega^\top, \Delta \eta_s^\top, \Delta \dot{\eta}_s^\top\}^\top$. The matrices E_r and A_r are given by

$$E_r = \begin{bmatrix} mI & -m\tilde{r}_{C,t} & 0 & m\Lambda \\ m\tilde{r}_{C,t} & I_{B,t}(\eta_t) & 0 & H(\eta_t) \\ 0 & 0 & I & 0 \\ m\Lambda^\top & H^\top(\eta_t) & 0 & M \end{bmatrix} \quad (65)$$

$$A_r = A_{r0} + \bar{q} \begin{bmatrix} \bar{C}_{\beta\beta} & \bar{C}_{\beta\eta} & \bar{C}_{\beta\dot{\eta}} \\ 0_{n \times (2n+6)} & & \\ \bar{C}_{\eta\beta} & \bar{C}_{\eta\eta} & \bar{C}_{\eta\dot{\eta}} \end{bmatrix} \quad (66)$$

where

The matrix B_r is given by

$$B_r = \begin{bmatrix} C_{\beta T} & \bar{q} \bar{C}_{\beta\delta} \\ 0_{n \times n_u} & \\ C_{\eta T} & \bar{q} \bar{C}_{\eta\delta} \end{bmatrix} \quad (68)$$

Assuming that a reduced-order linear state space model is available from model order reduction from a higher-order model, with A_{ref} , B_{ref} matrices, the aerodynamic coefficients are acquired through matching $E_r^{-1} A_r$ to A_{ref} , and matching $E_r^{-1} B_r$ to B_{ref} . Additional terms $C_{\beta 0}$ and $C_{\eta 0}$ are defined by the zero state time rate of change condition at the trim condition. This linearization formula from Eqs. (64–68) also helps in evaluating the modeling error in the residualization procedure due to discarding modes. All the residualization-related terms $\bar{C}_{\beta\eta}$, $\bar{C}_{\beta\eta}$, $\bar{C}_{\beta\dot{\eta}}$, $\bar{C}_{\beta\delta}$, $\bar{C}_{\eta\eta}$, $\bar{C}_{\eta\dot{\eta}}$, $\bar{C}_{\eta\ddot{\eta}}$, $\bar{C}_{\eta\dot{\eta}}$, $\bar{C}_{\eta\beta}$, and $\bar{C}_{\eta\delta}$ can be generated with different numbers of discarded modes, and the corresponding frequency-domain modeling error due to residualization can be generated by generating the frequency response function of Eq. (64).

IV. Simulation Case Studies

A. Case Studies with a General Transport Aircraft

The proposed semi-analytical FA model is numerically verified in this section for a general transport aircraft, which weighs 7139.74 kg and has a 19 m wingspan. The UM/NAST beam-based model is selected to be the high-order benchmark model, which is shown in Fig. 2. The detailed process of creating this model is described in [29]. The nonlinear UM/NAST model has 333 states, including 13 rigid states and 320 elastic states, considering the flexibility of the wings and the fuselage. The GTA weighs 7139.74 kg, and the aerodynamics linearization is acquired at a trimmed cruise flight at the altitude of 6096 m and a speed of 160 m/s. At this flight condition, the trimmed angle of attack is 1.345°.

Three representative test cases are designed to show the longitudinal, lateral, and coupled responses of the semi-analytical modeling of GTA. The excitations are defined as follows:

- 1) A longitudinal excitation from elevator doublet with 1 deg amplitude
- 2) A lateral excitation from aileron doublet with 1 deg amplitude
- 3) A coupled excitation of elevator and aileron with 1 deg amplitude

The doublet inputs are illustrated in Fig. 3. In the first two excitations, the doublets are introduced with $t_{\text{start}} = 1$ s, $t_{\text{half}} = 4$ s, and $t_{\text{end}} = 7$ s. In the coupled excitation, an identical elevator doublet profile is introduced, while the aileron doublet is introduced 1 s later, such that the contributions to the aircraft responses are more distinguishable. The translation velocities (v_x , v_y , and v_z), rotation velocities (p , q , and r), Euler angles (ϕ , θ , and ψ), and representative elastic states are shown in Figs. 4–6. To evaluate the aeroelastic behavior, four elastic modes ($n = 4$) are preserved in the proposed low-order semi-analytical model. The first out-of-plane bending modes at 1.48 Hz are most dominant in this configuration; thus, the amplitudes of these modes on the left and right wings (η_L and η_R) are selected to represent the responses of the flexible states. Under the linearity assumption, the unity values of η_L and η_R correspond to the wing tip upward deflection of 0.11 m.

The rigid-body and incremental out-of-plane bending modal responses under 1 deg elevator doublet input, using high-order UM/NAST and proposed semi-analytical modeling with/without residualization (in Sec. III.A), are shown in Fig. 4. The primary responses to this longitudinal excitation are the forward velocity v_x , vertical velocity v_z , pitch rate q , and pitch angle θ . The proposed semi-analytical model responses match the UM/NAST responses, especially with the proposed residualization. For the rest of the lateral responses, small deviations are observed for the proposed semi-analytical models. This is attributed to the significantly fewer numbers of modes (less than 10%) being used as compared to the high-order models and the small spill-over effects unavoidable with any model order reduction. Despite these small deviations, the out-of-plane bending modes $\Delta\eta_L$ and $\Delta\eta_R$ are very accurately captured. In the simulated descending maneuver, the bending modal amplitudes follow the doublet sequence, and the responses on the left and right wings are symmetrical.

The rigid-body and incremental out-of-plane bending modal responses under 1 deg aileron doublet input are shown in Fig. 5.

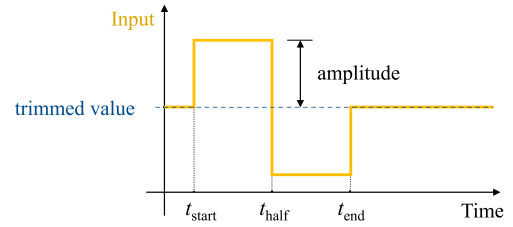


Fig. 3 Input doublet profile.

The primary responses to this input are the lateral velocity v_y , the roll rate p , the yaw rate r , the roll angle ϕ , and the yaw angle ψ . These lateral terms are quite off for the semi-analytical model without residualization; however, they are more accurately captured with residualization. Unlike the longitudinal responses in Fig. 4, the response for this lateral excitation introduces antisymmetrical modal responses on the left and right wings. Note that a slight constant deviation arises from the small trim difference between the proposed semi-analytical model and the high-order UM/NAST model, but the trend due to dynamics is captured. The residualization is observed to not only affect the elastic responses but also have a significant impact on the rigid-body responses. The out-of-plane bending terms $\Delta\eta_L$ and $\Delta\eta_R$ are very accurately captured. Unlike the longitudinal responses in Fig. 4, the response for this lateral excitation introduces antisymmetrical modal responses on the left and right wings.

The responses to 1 deg coupled doublet input are shown in Fig. 6, which further confirm that the semi-analytical model with residualization captures the dynamics of the high-order UM/NAST mode with significantly fewer states. This coupled excitation response has both longitudinal and lateral excitations to the FA. The longitudinal excitation mainly introduces lift and pitch moment to the aircraft, while the lateral excitation introduces mainly roll moment to the aircraft. This effect of coupled excitations is well observed in the simulated responses in Fig. 6. Also, the out-of-plane bending terms $\Delta\eta_L$ and $\Delta\eta_R$ are very accurately captured. The responses' major contribution arises from the longitudinal excitation as it introduces lift, but the characteristic drop of $\Delta\eta_R$ at around 5 s is also accurately captured, indicating the effectiveness of the semi-analytical model.

To further analyze the modeling error of the proposed semi-analytical method, the integrated square of the response error normalized by the total responses (away from the trim) is selected as a model accuracy metric. These normalized errors for the rigid-body states, flexible states, and all states of interests are defined as e_r , e_f , and e_{all} ; they are given by

$$\begin{aligned} e_r &= \frac{\int_0^{T_{\text{sim}}} |y_{r,\text{NAST}}(t) - y_{r,\text{ROM}}(t)|^2 dt}{\int_0^{T_{\text{sim}}} |y_{r,\text{NAST}}(t)|^2 dt}, \\ e_f &= \frac{\int_0^{T_{\text{sim}}} |y_{f,\text{NAST}}(t) - y_{f,\text{ROM}}(t)|^2 dt}{\int_0^{T_{\text{sim}}} |y_{f,\text{NAST}}(t)|^2 dt}, \\ e_{\text{all}} &= \frac{\int_0^{T_{\text{sim}}} |y_{\text{NAST}}(t) - y_{\text{ROM}}(t)|^2 dt}{\int_0^{T_{\text{sim}}} |y_{\text{NAST}}(t)|^2 dt} \end{aligned} \quad (69)$$

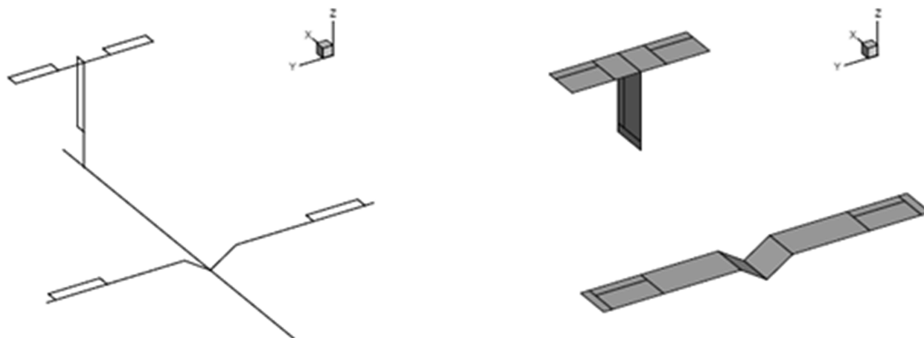


Fig. 2 General transport aircraft UM/NAST modeling [29].

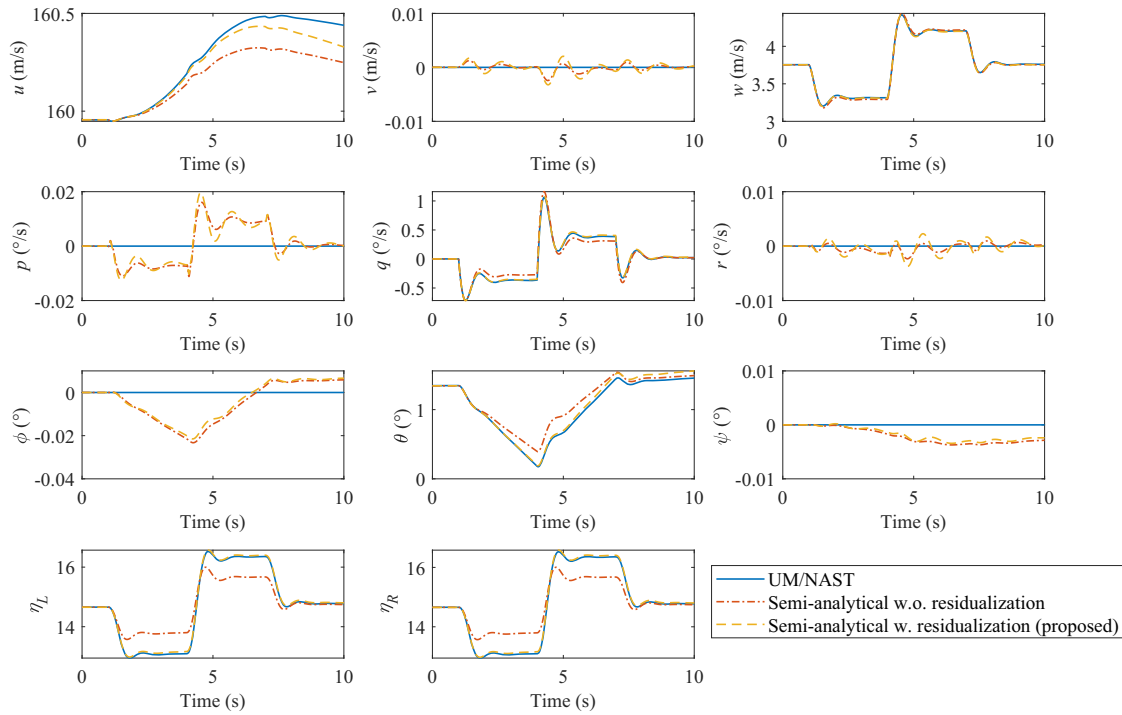


Fig. 4 Rigid-body and out-of-plane bending modal responses of GTA under 1 deg elevator doublet input.

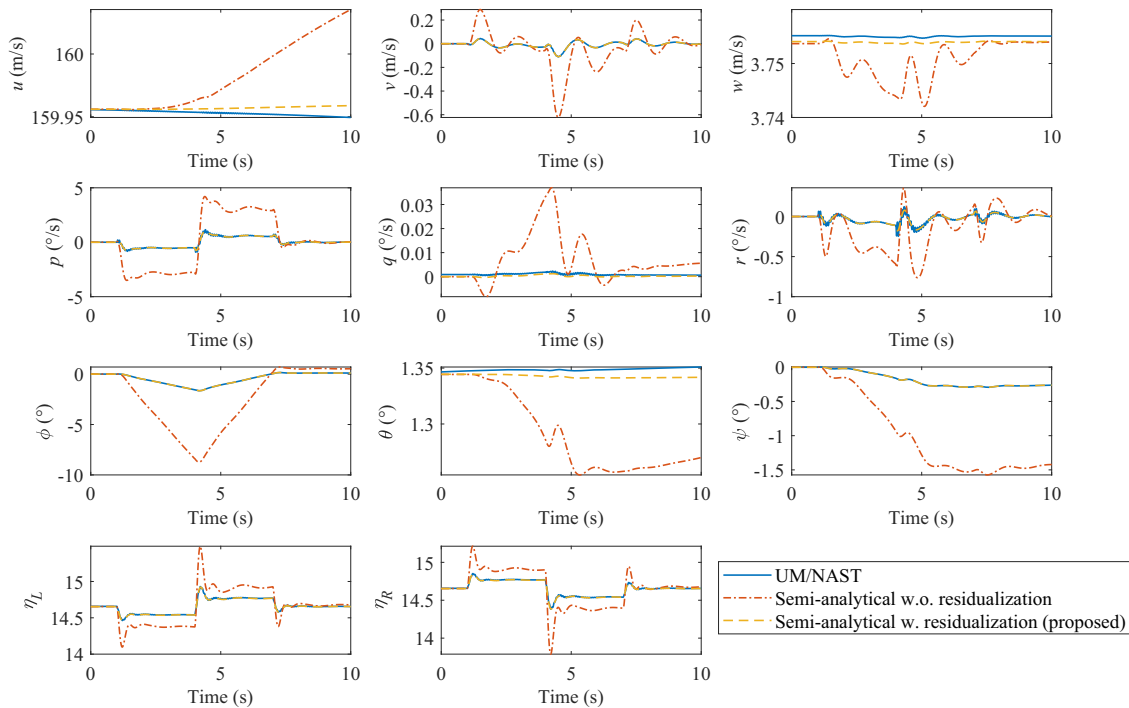


Fig. 5 Rigid-body and out-of-plane bending modal responses of GTA under 1 deg aileron doublet input.

where $T_{\text{sim}} = 10$ is the simulation time, while $y_{f,\text{NAST}}$ and $y_{f,\text{ROM}}$ are the rigid-body states of interest (i.e., $\Delta\beta$, $\Delta\phi$, $\Delta\theta$, and $\Delta\psi$) for UM/NAST and the proposed semi-analytical reduced-order model, respectively. The same notion applies to the flexible states ($y_{f,\text{NAST}}$ and $y_{f,\text{ROM}}$) and total states (y_{NAST} and y_{ROM}). The errors are evaluated based on the deviation of the responses from the trim condition, such that only the dynamic response differences are compared. Figure 7 illustrates the modeling error for the three different test cases with increasing excitation amplitudes. The presented percentage exaggerates the model inaccuracy in a way that the high-frequency oscillations in the UM/NAST model are intentionally

discarded in the semi-analytical model, while these high-frequency errors contribute to the total error integration in Eq. (69). The lateral excitation results in a relatively low modeling error (below 6%), while the longitudinal and coupled excitations exhibit an increasing modeling error with increasing amplitudes. This is attributed to the fact of increased geometrical nonlinearity, which is not captured in our FA modeling framework. In the maximum amplitudes shown in Fig. 7, the tip deflection reaches about 3.7 m, which is more than one-third of the total wingspan. Even with this large deformation violating the linear assumption, the proposed semi-analytical model still captures more than 80% of the dynamic response.

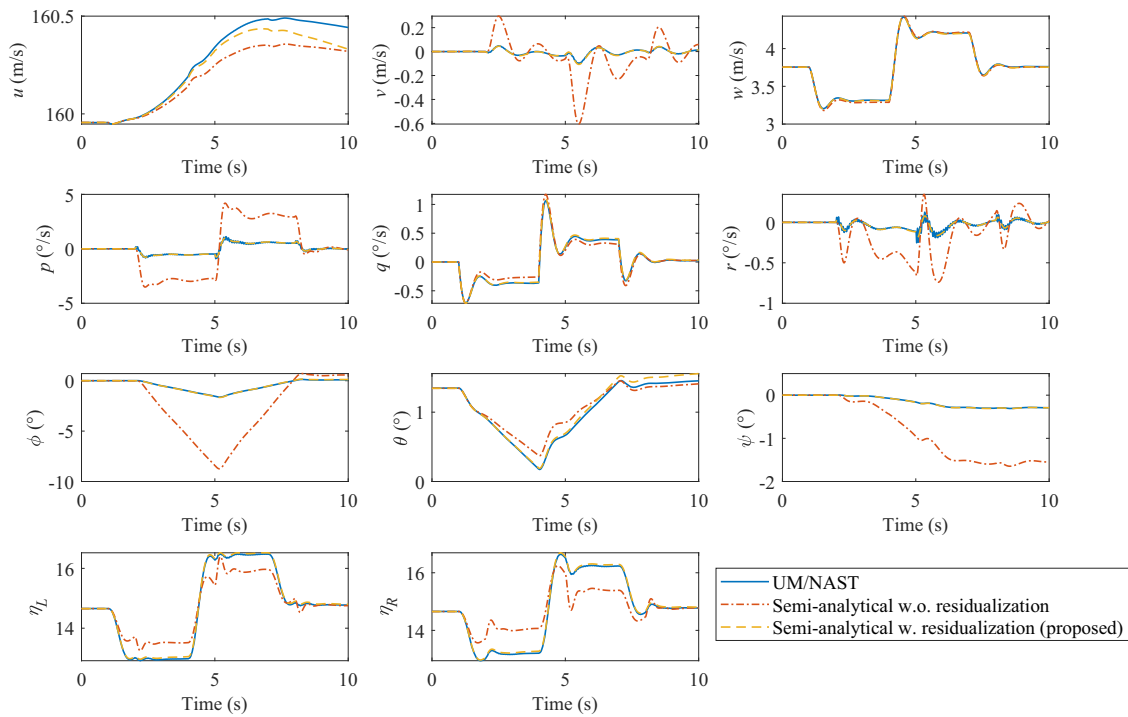


Fig. 6 Rigid-body and out-of-plane bending modal responses of GTA under 1 deg coupled doublet input.

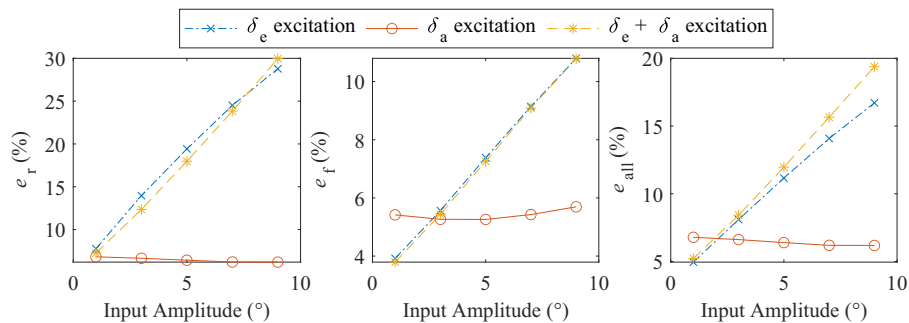


Fig. 7 Normalized error of the rigid-body states, flexible states, and all states for the longitudinal excitation, lateral excitation, and the coupled excitation for different amplitudes.

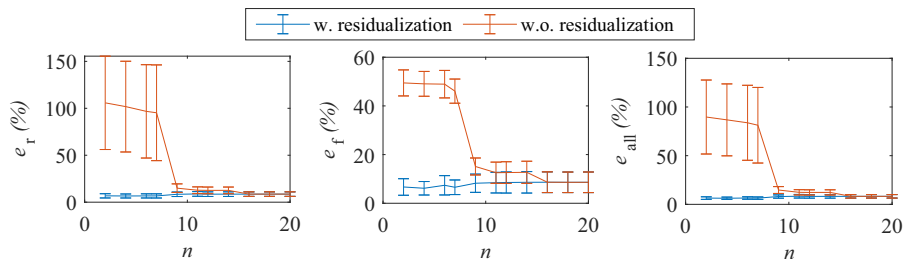


Fig. 8 Normalized error of the rigid-body states, flexible states, and all states for different number of preserved elastic modes of GTA, with and without residualization.

To understand the effect of the number of elastic states, random coupled doublet excitations from the elevators, ailerons, rudder, and thrust input are generated, and the model accuracy is assessed. The thrust forces are kept to have maximum amplitudes of 1000 N, and the control surface deflections have a maximum of 1 deg. The statistics of the modeling error of the rigid-body states, the flexible states, and all the states are shown in Fig. 8. Note that only the statistics for $n = 2, 4, 6, 7, 9, 11, 12, 14, 16, 18$, and 20 are presented. This selection is rationalized by the fact that the 7th and 11th modes are elastic modes of the fuselage and that the symmetric modes of the

left and right wings are always preserved in the proposed semi-analytical approach. In this GTA setup, the first two bending modes are dominant, such that considering only two elastic states yields relatively accurate results with residualization. However, the same level of modeling accuracy requires almost 18 elastic states without residualization. Also, the modeling error uncertainty is reduced with the proposed residualization.

As discussed in Sec. III.B, frequency-domain system representations can be used to analyze the modeling error due to the residualization procedure. This has been verified by comparing the frequency

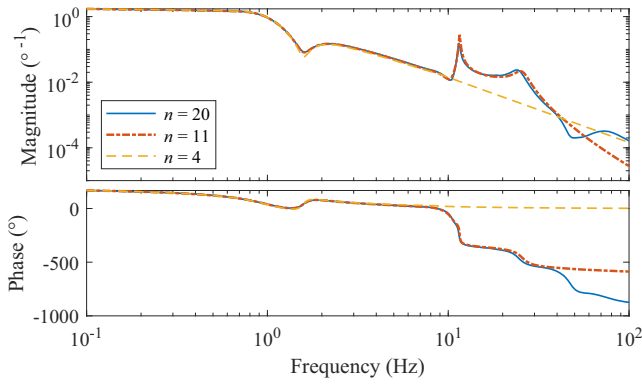


Fig. 9 Frequency-domain responses of GTA's elevator input to the first bending mode amplitudes, with residualization keeping different numbers of flexible modes.

response function (FRF) of FA from the joint elevator incremental input in degrees to the flexible responses. The FRFs with 20, 11, and 4 kept modes with corresponding cutoff frequencies at 63.19, 20.85, and 8.13 Hz, respectively, are shown in Fig. 9. The effect of residualization is clearly shown, as the low-frequency portions of the responses are kept the same due to the consideration of DC component preservation in the residualization procedure. The frequency ranges in which the proposed FA model has relatively high accuracy can be well estimated from the modal decomposition information. It

is also shown that the cutoff frequency is a desirable indicator of the frequencies beyond which significant deviation exists.

B. Case Studies with XRF1 Flexible Aircraft

In this section, the proposed semi-analytical approach is applied to a representative commercial transport aircraft. The XRF1 is an Airbus-provided industrial standard multidisciplinary research test case representing a typical configuration for a long-range, wide-body aircraft. The UM/NAST model of the XRF1 [30] shown in Fig. 10 is used as a reference full-order solution. The nonlinear UM/NAST model of the XRF1 has 1021 states, including 13 rigid states and 1008 elastic states. The XRF1 weighs 7139.74 kg, and it is trimmed at an altitude of 8484.108 m and a speed of 253.9825 m/s (i.e., Mach 0.83). The trimmed angle of attack is 1.315°.

The responses to the coupled excitation (third test case as in Sec. IV.A) of 1 deg amplitude using 10 elastic states are shown in Fig. 11. Clearly, the proposed semi-analytical model captures the dynamics with residualization. In this descending turn maneuver, the roll rate and angle are very accurately captured. The pitch and yaw responses have a slight error, but the trend is mostly captured. The modeling errors for different numbers of elastic modes are shown in Fig. 12. The modeling error is higher in the case of XRF1 than GTA since the XRF1 responses contain more high-frequency components, which exaggerate the error metrics. Also, more states are discarded in the XRF1 model, such that the modeling error is higher than in the case of the GTA model. In particular, having more than 14 states can further enhance the model's accuracy both with and without

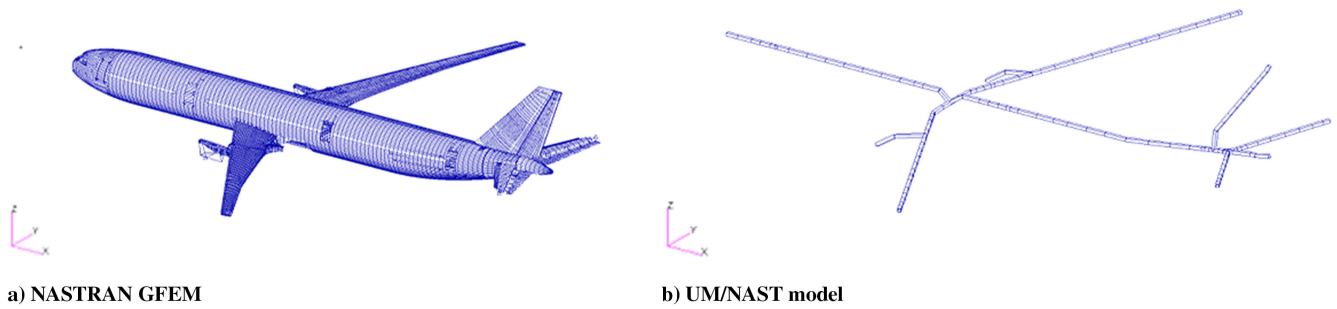


Fig. 10 XRF1 NASTRAN GFEM model and UM/NAST model [30].

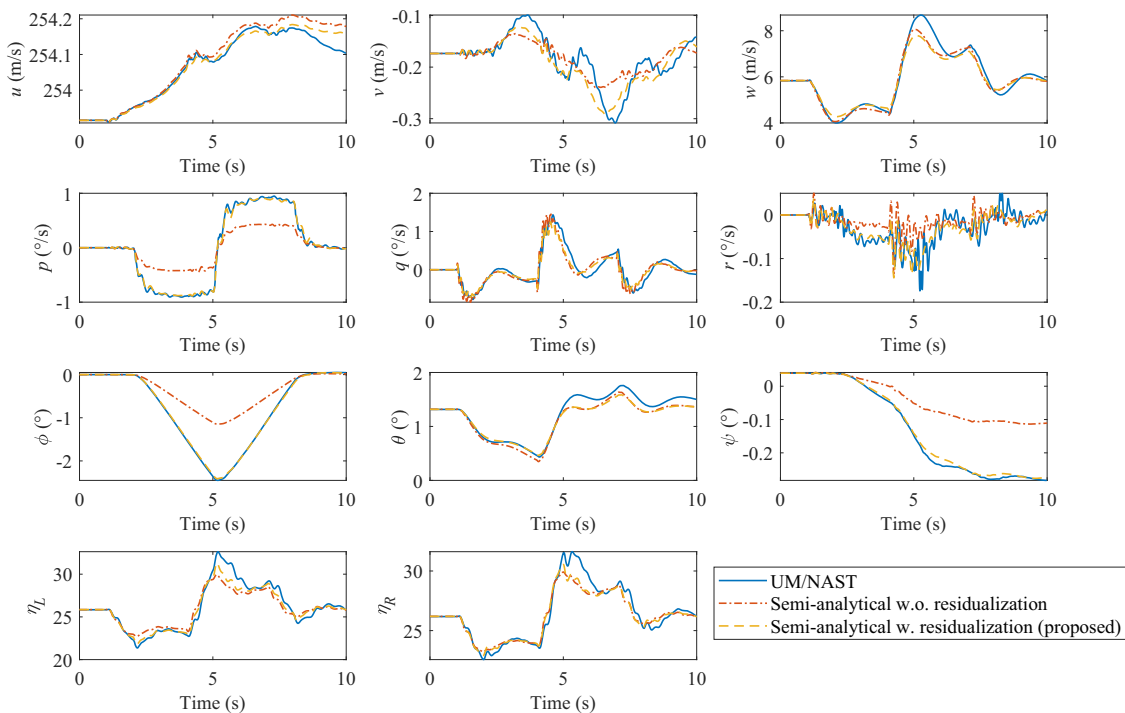


Fig. 11 Rigid-body and out-of-plane bending modal responses of XRF1 under 1 deg coupled doublet input.

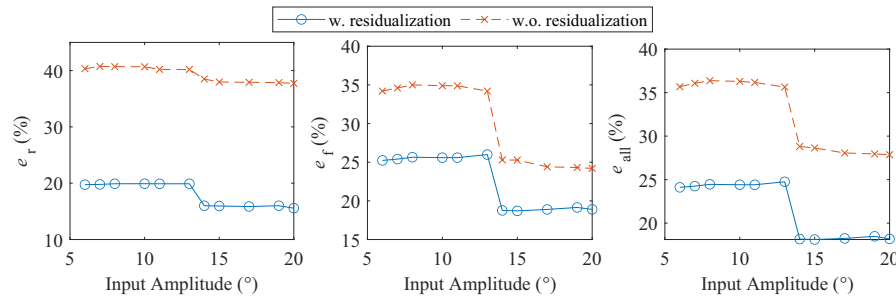


Fig. 12 Normalized error of the rigid-body states, flexible states, and all states for different number of preserved elastic modes of XRF1, with and without residualization.

residualization. Also, the residualization reduces the total modeling error by about 30%.

V. Conclusions

This paper introduced a semi-analytical low-order model of the flight dynamics of FA, where the geometric displacements, although in the linear range, cannot be accurately captured only by a flex-to-rigid ratio correction of aerodynamic effects in the 6-rigid-DOF equations of motion. It adopted the body-fixed axis formulation, and the FA dynamics were derived, including rigid and flexible degrees of freedom and preserving the physical interpretation of the various parameters in the problem. Aerodynamic linearization from high-order models was employed to enhance the modeling accuracy. To enhance the model's accuracy with fewer states, static residualization was also derived and adopted. In addition, the analytical linearization of the model was derived, as well as methods to extract the aerodynamic linearization from the state-space linear-time-invariant system realization. The new approach was verified for modeling a general transport aircraft and an Airbus XRF1 aircraft. Both longitudinal, lateral, and coupled dynamics were well captured, with significantly less number of states in comparison to reference high-order nonlinear UM/NAST models.

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